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Uncoupling of General Linear Multi-Degree-of-Freedom Structural and Mechanical Systems Through Quasi-Diagonalization

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This article develops a fundamental result in linear algebra by providing the necessary and sufficient conditions for the simultaneous quasi-diagonalization of two symmetric matrices and two skew-symmetric matrices by a real orthogonal congruence. This result is used to study the uncoupling of general linear multi-degree-of-freedom (MDOF) structural and mechanical systems described by arbitrary damping and stiffness matrices through quasi-diagonalization, and real orthogonal coordinate transformations. The uncoupling leads to independent subsystems, each having at most two degrees-of-freedom with a specific structure. The results encompass the different physical categories of linear MDOF systems identified by engineers, mathematicians, and physicists and provide the necessary and sufficient conditions for their maximal uncoupling. A total of 16 conditions are shown to exist. However, the number of such conditions for physical systems that are commonly met in nature as well as in aerospace, civil, and mechanical engineering are shown to be considerably less, dwindling at times to two or three, thereby making the results applicable to numerous high-order real-life linear MDOF dynamical systems. Several new analytical results are obtained and corroborated through numerical examples.
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1 Introduction

The experimental and analytical studies of the Bernoullis (Johann and Daniel) and Leonard Euler on linear multi-degree-of-freedom (MDOF) potential systems led to the concepts of natural frequencies of vibration, normal modes of vibration, and their superposition. Based on the well-known theorem in linear algebra—namely, that the necessary and sufficient (n&s) condition for a real orthogonal matrix to exist so that two real symmetric matrices can be simultaneously diagonalized is that they commute—Caughey and O'Kelly in 1965 extended the use of normal mode analysis to a damped MDOF potential system whose symmetric damping matrix commutes with its symmetric stiffness (potential) matrix [1]. This led to the uncoupling of such damped MDOF systems into independent single-degree-of-freedom subsystems through the use of a simple real orthogonal transformation. Geometrically, orthogonal transformations are simply rotations and/or reflections. The uncoupling provided deeper physical insights into the system's dynamical behavior and proffered robust methods for computing its free and forced response. In the following decades, this made modeling damped MDOF structural or mechanical potential systems, using a symmetric damping matrix that commutes with its stiffness matrix, somewhat of a norm in much of engineering research and practice when dealing with damped potential systems.

Like Caughey and O'Kelly used the theorem in linear algebra mentioned above for the uncoupling of damped potential systems into low-dimensional independent subsystems, the uncoupling of other categories of linear MDOF systems into low-dimensional independent subsystems through the use of real orthogonal coordinate transformations had to wait for further advances in linear algebra. For example, one needed to find the necessary and sufficient (n&s) conditions for a real orthogonal matrix to exist so that a skew-symmetric and a symmetric matrix could be simultaneously quasi-diagonalized (a term we explain later) to know if (when) a

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gyroscopic potential system can be uncoupled into lower-dimensional subsystems using a real linear orthogonal coordinate transformation. In Refs. [2,3], these n&s conditions are obtained and it is shown that gyroscopic potential systems can be uncoupled into at most two degrees-of-freedom independent subsystems when the appropriate n&s conditions are met. Considering other categories of structural and mechanical systems, Ref. [4] extends these results to damped gyroscopic MDOF potential systems wherein the damping matrix is restricted to be symmetric. Noting that many real-life potential systems may not have symmetric damping matrices, and even if they did, their damping matrices may not commute with their stiffness matrices, a different category of MDOF potential systems that have arbitrary damping matrices are considered in Ref. [5], and the n&s conditions for such systems to be uncoupled through quasi-diagonalization into independent subsystems, each of at most two degrees-of-freedom, are obtained. Reference [6] deals with developing the n&s conditions for uncoupling MDOF gyroscopic systems with arbitrary stiffness matrices through quasi-diagonalization, yielding independent subsystems with at most two degrees-of-freedom; this includes the important class of nonconservative systems with positional circulatory forces. Reference [7] considers the quasi-diagonalization and uncoupling of MDOF systems with symmetric damping matrices subjected to positional nonconservative forces. This article provides a unified approach for obtaining the n&s conditions for the maximal uncoupling of such systems and many others, modeled by linear MDOF systems that commonly arise in aerospace, civil, and mechanical engineering, as well as in nature.

The most general linear MDOF mechanical system is one in which both the damping matrix and the stiffness matrix are arbitrary, and it is these systems that this article addresses. We consider the general nonconservative system described by the equation

$$\tilde{M}\ddot{q} + \tilde{C}\dot{q} + \tilde{R}q = \tilde{f}(t) \tag{1}$$

where $q(t)$ and $\tilde{f}(t)$ are n-vectors (n by 1 column vectors), $M^T = M > 0$, and $\tilde{R}$ and $\tilde{C}$ are arbitrary constant matrices that provide the position-dependent and velocity-dependent forces, respectively. The real matrices $\tilde{M}$, $\tilde{C}$, and $\tilde{R}$ are each n by n matrices, and the dots indicate differentiation with respect to time, t. We shall assume throughout this article that the mass matrix, $\tilde{M}$, is positive definite.

Using the real transformation $q(t) = \tilde{M}^{-1/2}x(t)$, where $\tilde{M}^{-1/2}$ denotes the inverse of the unique positive-definite square root of $\tilde{M}$, (1) reduces to the relation

$$\ddot{x} + D\dot{x} + Rx = f(t) \tag{2}$$

where

$$D = \tilde{M}^{-1/2}\tilde{C}\tilde{M}^{-1/2} \tag{3}$$

$$R = \tilde{M}^{-1/2}\tilde{R}\tilde{M}^{-1/2} \tag{4}$$

and

$$f(t) = \tilde{M}^{-1/2}\tilde{f}(t) \tag{5}$$

We shall refer to the arbitrary n by n matrices D and R as the damping and stiffness matrices, respectively, and the n-vector $f(t)$ as the force.

Lemma 1. *The real matrices $\tilde{C}$ and $\tilde{R}$ in (1) can each be (uniquely) split into the sum of two n by n matrices, one of which is symmetric and the other skew-symmetric. Likewise, the matrices D and R.*

Proof. See, for example, Ref. [8]. ∎

The matrices $\tilde{C}$ and $\tilde{R}$ in (1) can be uniquely split as

$$\tilde{C} = \frac{\tilde{C}+\tilde{C}^T}{2} + \frac{\tilde{C}-\tilde{C}^T}{2} := \tilde{S} + \tilde{G} \tag{6}$$

and

$$\tilde{R} = \frac{\tilde{R}+\tilde{R}^T}{2} + \frac{\tilde{R}-\tilde{R}^T}{2} = \tilde{K} + \tilde{N} \tag{7}$$

where the symmetric parts of the matrices $\tilde{C}$ and $\tilde{R}$ are denoted by the symmetric matrices $\tilde{S}$ and $\tilde{K}$, respectively, and their skew-symmetric parts are denoted by $\tilde{G}$ and $\tilde{N}$, respectively. Using (6) and (7), (1) can then be rewritten as

$$\tilde{M}\ddot{q} + (\tilde{S}+\tilde{G})\dot{q} + (\tilde{K}+\tilde{N})q = \tilde{f}(t) \tag{8}$$

Similarly, the matrices D and R in (2) can be split as

$$D = \frac{D+D^T}{2} + \frac{D-D^T}{2} := S + G \tag{9}$$

and

$$R = \frac{R+R^T}{2} + \frac{R-R^T}{2} := K + N \tag{10}$$

where we have denoted the symmetric part of D by S and the skew-symmetric part of D by G, as well as the symmetric part of R by K and the skew-symmetric part of R by N.

Using (9) and (6) in (3), and (10) and (7) in (4) gives

$$\begin{aligned} &S = \tilde{M}^{-1/2}\tilde{S}\tilde{M}^{-1/2},\ G = \tilde{M}^{-1/2}\tilde{G}\tilde{M}^{-1/2}, \\ &K = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2},\ N = \tilde{M}^{-1/2}\tilde{N}\tilde{M}^{-1/2} \end{aligned} \tag{11}$$

Also, from (9) and (10), (2) can be written as

$$\ddot{x} + \underbrace{(S+G)}_{D}\dot{x} + \underbrace{(K+N)}_{R}x = f(t) \tag{12}$$

This equation is equivalent to (1) and we will be primarily using it in what follows. Furthermore, in this article, the n by n matrices S and K are always taken to be symmetric, and the n by n matrices G and N are always taken to be skew-symmetric. Throughout this article, when the n by n skew-symmetric matrix $G(N)$ has rank $2m > 0$, we will often denote this rank condition simply as $G(N) \neq 0$, for short. Note that this short-form notation, besides saying that $G(N)$ is a nonzero matrix, also includes its rank as being $2m$. These matrices G and N always refer to the skew-symmetric (additive) parts of D and R, respectively, while S and K always refer to the symmetric (additive) parts of D and R, respectively. In the literature, the matrix N is often referred to as a circulatory matrix, and $-Nx$ as the positional circulatory force.

We note that by setting the appropriate matrices to zero, (12) encompasses several different categories of linear MDOF systems commonly encountered in nature and in engineered systems [9]. Some examples are:

(i) Conservative potential systems ($S = N = G = 0$).
(ii) Gyroscopic potential systems ($S = N = 0$).
(iii) Gyroscopic nonconservative systems ($S = 0$).
(iv) Gyroscopic circulatory systems ($S = 0$, $K = 0$).
(v) Damped potential systems with symmetric damping matrices ($G = 0$, $N = 0$).
(vi) Damped potential systems with arbitrary damping matrices ($N = 0$).
(vii) Damped gyroscopic potential systems with symmetric damping matrices ($N = 0$).
(viii) Damped system with an arbitrary damping matrix and a purely circulatory positional force ($K = 0$).
(ix) Damped gyroscopic system with only a purely circulatory positional force and a symmetric damping matrix ($K = 0$).

(x) General nonconservative systems with velocity-dependent damping forces and positional circulatory forces $(S, G, K, N \neq 0)$.

As noted in Ref. [5], though the MDOF dynamical systems described in (vi) and (vii) above have the same mathematical structure and may be thought as duals of one another, the character of the physical forces that engender them is widely different. Likewise, the dynamical systems (viii) and (ix), which may also be considered duals of one another, are described by equations that have the same structure. However, the physical forces that engender them are widely different in character and origin.

Our overall goal is to obtain the n&s conditions for uncoupling the MDOF system described by (12) through quasi-diagonalization so that a real change of coordinates $x = Qp$, where Q is a real orthogonal matrix, transforms it into a canonical (simplest) form that is maximally uncoupled. We shall refer to p as the principal coordinate.

If we assume that such a real orthogonal matrix Q exists, upon multiplication of (12) from the left by Q^T and the use of this coordinate change, we would obtain

$$\ddot{p} + Q^T(S+G)Q\dot{p} + Q^T(K+N)Qp = Q^T f(t) \tag{13}$$

so that our aim, from a linear algebra standpoint, then becomes finding Q such that the matrices Q^TSQ, Q^TGQ, Q^TKQ, and Q^TNQ are each direct sums with each diagonal block having a minimal size.

In what follows, we will show that a real orthogonal matrix Q can be found such that the dynamical system (12) can be uncoupled into independent subsystems, each of which has no more than two degrees-of-freedom when the matrices S, G, K, and N satisfy certain necessary and sufficient conditions. These necessary and sufficient (n&s) conditions, under which this uncoupling is guaranteed through a quasi-diagonalization of these four matrices, are explicitly obtained in terms of commutators that involve them. We first present and prove several preliminary theorems upon which our results will rest.

The structure of this article is as follows. In Sec. 2, we present the central theorems in linear algebra that obtain the n&s conditions for the simultaneous quasi-diagonalization of two real symmetric and two real skew-symmetric matrices using a real orthogonal congruence. Particularizations of the central theorems when one of the four matrices is zero are provided. Section 3 uses these results to provide the n&s conditions in which systems described by (12) that have arbitrary damping matrices D and arbitrary stiffness matrices R can be uncoupled into independent subsystems that have at most two degrees-of-freedom; a total of 16 conditions are obtained. Section 3.1 deals with uncoupling various categories [9] of linear MDOF systems in a unified manner. It shows the generality of the n&s conditions obtained herein and their easy application to different categories of structural and mechanical systems. Depending on the ranks of the skew-symmetric matrices G and N, commonly found in real-life applications, the n&s conditions are shown to drop to ten in number. In Sec. 3.2, we obtain further reductions in the number of n&s conditions. For example, when the nonzero eigenvalues of the skew-symmetric matrices G and N are distinct, a common occurrence in many physical systems, the number of n&s can drop to six, and then to four depending on the ranks of G and N. Further reductions down to just two n&s conditions are shown to result when K and N have posited form. Section 4 gives the conclusions.

2 Fundamental Theorems

We begin by stating the four new fundamental theorems in linear algebra that will be used later.

THEOREM 1. *Given the n by n matrices G, N, K, and S, with $0 < Rank(G) = 2m \le n (G \neq 0)$, there exists a real orthogonal matrix Q such that*

$$\Gamma = Q^TGQ = \mathrm{diag}\,(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m}) \tag{14}$$

$$\begin{aligned} \mathrm{N} = Q^TNQ &= \mathrm{diag}\,(\nu_1 J_2, \ldots, \nu_m J_2, \ldots, \nu_{n/2} J_2) \text{ for } n \text{ even} \\ &= \mathrm{diag}\,(\nu_1 J_2, \ldots, \nu_m J_2, \ldots, \nu_{(n-1)/2} J_{(n-1)/2}, 0) \text{ for } n \text{ odd} \end{aligned} \tag{15}$$

$$\Lambda = Q^TKQ = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{16}$$

and

$$\Sigma = Q^TSQ = \mathrm{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{17}$$

where $\beta_j > 0$, $j = 1, \ldots, m$, and all the ν_j's, λ_j's, and σ_j's are real numbers if and only if the following set of conditions are satisfied:

$$\begin{aligned} &[G, N] = 0, && [K, S] = 0, && [K, GN] = 0, && [S, GN] = 0, \\ &[K, GKG] = 0, && [K, GSG] = 0, && [K, NKN] = 0, && [K, NSN] = 0, \\ &[S, GKG] = 0, && [S, GSG] = 0, && [S, NKN] = 0, && [S, NSN] = 0, \\ &[K, G^2] = 0, && [S, G^2] = 0, && [K, N^2] = 0, && [S, N^2] = 0 \end{aligned} \tag{18}$$

In (14) and (15), $J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and the matrices Γ and N are called "quasi-diagonal"; the difference in symbols between the quasi-diagonal matrix, N, *and the arbitrary skew-symmetric matrix, N, should be noted. In (18), we have used the commutator notation in which, given any two n by n matrices A and B, the commutator* $[A, B] := AB - BA$. ■

We denote a diagonal matrix by diag(*), where "*" gives its ordered diagonal elements.

When the matrices G, N, K, and S can be reduced to the forms (14)–(17) by an orthogonal congruence, we shall say that these matrices are *simultaneously quasi-diagonalized by the real orthogonal matrix Q* or *simultaneously orthogonally quasi-diagonalized*, for short.

Remark 1. Equation (14) shows that the matrices Γ and G are related by an orthogonal congruence; therefore, they have the same eigenvalues. Noting that Γ is block diagonal, the n eigenvalues of G are then $\{\pm i\beta_1, \pm i\beta_2, \ldots, \pm i\beta_m, 0, \ldots, 0\}$, with $\beta_i > 0$, $i = 1, 2, \ldots, m$. Similarly, (15) shows that the eigenvalues of N are $\{\pm i\nu_1, \pm i\nu_2, \ldots, \pm i\nu_{n/2}\}$ when n is even, and $\{\pm i\nu_1, \pm i\nu_2, \ldots, \pm i\nu_{(n-1)/2}, 0\}$ when n is odd. Likewise, the λ_j's in the diagonal matrix Λ shown in (16) are the eigenvalues of K, and the σ_j's in the diagonal matrix Σ shown in (17) are the eigenvalues of S.

The next theorem restricts the rank, $2m$, of the matrix G so that $2m \ge n-2$. This is equivalent to saying that the dimension n (of the n by n matrices S, G, K, and N) exceeds the rank, $2m$, of the matrix G by at most 2, or, that $n \le 2m+2$. Hence, $2m \le n \le 2m+2$.

THEOREM 2. *Given the n by n matrices G, N, K, and S, with $0 < Rank(G) = 2m \le n (G \neq 0)$ and $n \le 2m+2$, there exists a real orthogonal matrix Q such that*

$$\begin{aligned} \Gamma = Q^TGQ &= diag\,(\beta_1 J_2, \ldots, \beta_m J_2) \text{ for } n = 2m \\ &= diag\,(\beta_1 J_2, \ldots, \beta_m J_2, 0) \text{ for } n = 2m+1 \\ &= diag\,((\beta_1 J_2, \ldots, \beta_m J_2, 0, 0) \text{ for } n = 2m+2 \end{aligned} \tag{19}$$

$$\begin{aligned} \mathrm{N} = Q^TNQ &= diag\,(\nu_1 J_2, \ldots, \nu_m J_2) \text{ for } n = 2m \\ &= diag\,(\nu_1 J_2, \ldots, \nu_m J_2, 0) \text{ for } n = 2m+1 \\ &= diag\,(\nu_1 J_2, \ldots, \nu_m J_2, \nu_{m+1} J_2) \text{ for } n = 2m+2 \end{aligned} \tag{20}$$

$$\Lambda = Q^TKQ = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{21}$$

and

$$\Sigma = Q^T SQ = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{22}$$

where $\beta_j > 0$, $j = 1, \ldots, m$, and all the ν_j's, λ_j's, and σ_j's are real numbers if and only if the following set of conditions are satisfied:

$$[G, N] = 0, \quad [K, S] = 0, \quad [K, GN] = 0, \quad [S, GN] = 0 \tag{23}$$

$$[K, GKG] = 0, \quad [K, GSG] = 0, \quad [S, GKG] = 0, \quad [S, GSG] = 0 \tag{24}$$

$$[K, G^2] = 0, \quad [S, G^2] = 0 \tag{25}$$

■

Remark 2. The set of commutators in (18) in Theorem 1 remains unchanged when the matrices G and N are interchanged and/or when the matrices K and S are interchanged. Also, the set of commutators in (23)–(25) remain unchanged when the matrices K and S are interchanged, but they change when G and N are interchanged.

Interchanging the roles of G and N in Theorem 1 and noting Remark 2, we have the following theorem.

THEOREM 3. *Given the n by n matrices G, N, K, and S, with $0 < Rank(N) = 2m \le n (N \ne 0)$, there exists a real orthogonal matrix Q such that*

$$\begin{aligned}\Gamma = Q^T GQ &= \text{diag}(\beta_1 J_2, \ldots, \beta_m J_2, \ldots, \beta_{n/2} J_2) \quad \text{for } n \text{ even} \\ &= \text{diag}(\beta_1 J_2, \ldots, \beta_m J_2, \ldots, \beta_{(n-1)/2} J_{(n-1)/2}, 0) \quad \text{for } n \text{ odd}\end{aligned} \tag{26}$$

$$\mathrm{N} = Q^T NQ = \text{diag}(\nu_1 J_2, \ldots, \nu_m J_2, 0_{n-2m}) \tag{27}$$

$$\Lambda = Q^T KQ = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{28}$$

and

$$\Sigma = Q^T SQ = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{29}$$

where $\nu_j > 0$, $j = 1, \ldots, m$, and all the β_j's, λ_j's, and σ_j's are real numbers, if and only if the set of commutation conditions (18) are satisfied. The n eigenvalues of N are $\{\pm i\nu_1, \pm i\nu_2, \ldots, \pm i\nu_m, 0, \ldots, 0\}$.

Proof. Interchanging the roles of G and N in Theorem 1, we obtain (26)–(29). Using Remark 2, the n&s conditions remain unchanged. ■

Remark 3. It is important to distinguish the situation when $0 < Rank(G) = 2m \le n$ (recall, $G \ne 0$, for short) and when the matrix G is the zero matrix ($G = 0$). Theorem 1 deals with the matrices G, N, K, and S, in which $0 < Rank(G) = 2m \le n$ ($G \ne 0$); it covers, of course, the situation when $N = 0$ (i.e., N is the zero matrix). Thus, when $G \ne 0$, application of Theorem 1 gives the n&s conditions, (18), for the simultaneous orthogonal quasi-diagonalization of the four matrices $G \ne 0$, N, K, and S. On the other hand, Theorem 3 deals with the same four matrices except that now $0 < Rank(N) = 2m \le n$ ($N \ne 0$); it covers the situation when $G = 0$ (i.e., G is the zero matrix). Therefore, the n&s conditions for the simultaneous orthogonal quasi-diagonalization of the matrices $G = 0$, N, K, and S, with $0 < Rank(N) = 2m \le n$ ($N \ne 0$), can be obtained by applying Theorem 3 and setting $G = 0$, again, in (18), since the two theorems share the same set of n&s conditions. Thus we observe that, in effect, the set (18) of n&s conditions can be used whether or not $G = 0$. The only difference between the situation when the matrix $G \ne 0$ and $G = 0$ lies in the quasi-diagonal forms that emerge from the resulting simultaneous quasi-diagonalization engendered by the orthogonal matrix Q. When $G \ne 0$, we use Theorem 1, and the quasi-diagonal forms in (14)–(17) emerge; when $G = 0$, we use Theorem 3, and the quasi-diagonal forms described in (26)–(29) emerge, with $\Gamma = 0$ since $G = 0$.

Interchanging the roles again of G and N in Theorem 2, we get the following result.

THEOREM 4. *Given the n by n matrices G, N, K, and S, with $0 < Rank(N) = 2m \le n$ ($N \ne 0$), and $n \le 2m + 2$, there exists a real orthogonal matrix Q such that*

$$\begin{aligned}\Gamma = Q^T GQ &= diag(\beta_1 J_2, \ldots, \beta_m J_2) \textit{ for } n = 2m \\ &= diag(\beta_1 J_2, \ldots, \beta_m J_2, 0) \textit{ for } n = 2m + 1 \\ &= diag(\beta_1 J_2, \ldots, \beta_m J_2, \beta_{m+1} J_2) \textit{ for } n = 2m + 2\end{aligned} \tag{30}$$

$$\begin{aligned}\mathrm{N} = Q^T NQ &= diag(\nu_1 J_2, \ldots, \nu_m J_2) \textit{ for } n = 2m \\ &= diag(\nu_1 J_2, \ldots, \nu_m J_2, 0) \textit{ for } n = 2m + 1 \\ &= diag((\nu_1 J_2, \ldots, \nu_m J_2, 0, 0) \textit{ for } n = 2m + 2\end{aligned} \tag{31}$$

$$\Lambda = Q^T KQ = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{32}$$

and

$$\Sigma = Q^T SQ = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{33}$$

where $\nu_j > 0$, $j = 1, \ldots, m$, and all the β_j's, λ_j's, and σ_j's are real numbers if and only if the following set of conditions are satisfied:

$$[G, N] = 0, \quad [K, S] = 0, \quad [K, GN] = 0, \quad [S, GN] = 0 \tag{34}$$

$$[K, NKN] = 0, \quad [K, NSN] = 0, \quad [S, NKN] = 0, \quad [S, NSN] = 0 \tag{35}$$

$$[K, N^2] = 0, \quad [S, N^2] = 0 \tag{36}$$

■

The proofs of the theorems provide an explicit way to construct the orthogonal matrix Q. Results for the simultaneous orthogonal quasi-diagonalization of three of the four matrices, for example, G, S, and K (the matrix triple $\{G, S, K\}$), using the central theorems are also obtained in this section.

We first prove Theorems 2 and 4, and later prove Theorems 1 and 3, which are more general. To improve the clarity of the exposition, we prove the sufficiency and the necessity of the commutation conditions in each theorem separately since they require several auxiliary results. These auxiliary theorems and lemmas are provided along the way.

Remark 4. The following five properties of commutators will be useful in this article.

1. From the definition of the commutator, we see that the n by n zero matrix commutes with all n by n matrices so that $[0, B] = [A, 0] = 0$. Also, $[A, A] = [A^2, A] = [A, A^2] = 0$.
2. $[A, B] = 0$ implies that $[B, A] = 0$, since $[B, A] = -[A, B]$.
3. Two diagonal matrices always commute, and so the commutator of the two diagonal matrices Λ_1 and Λ_2 yields $[\Lambda_1, \Lambda_2] = [\Lambda_1\Lambda_2 - \Lambda_2\Lambda_1] = 0$. Consider the matrix $A = aQ\Lambda_1 Q^T$, where a is a scalar, Q is an orthogonal matrix, and Λ_1 is a diagonal matrix; likewise, the matrix $B = bQ\Lambda_2 Q^T$, where b is a scalar and Λ_2 is diagonal. Then the commutator $[A, B] = ab[Q\Lambda_1 Q^T, Q\Lambda_2 Q^T] = ab[Q\Lambda_1 Q^T Q\Lambda_2 Q^T - Q\Lambda_2 Q^T Q\Lambda_1 Q^T] = abQ[\Lambda_1\Lambda_2 - \Lambda_2\Lambda_1]Q^T = 0$. The third equality follows from $Q^T Q = I$, and last equality follows because two diagonal matrices always commute.

(4) The commutators have the following two properties: $[AB, C] = A[B, C] + [A, C]B$ and $[A, BC] = [A, B]C + B[A, C]$. Using the first relation, if $[A, B] = 0$, then $[A^2, B] = [AA, B] = A[A, B] + [A, B]A = 0$, and similarly, if $[A, B] = 0$, then $[A, B^2] = 0$.

(5) If $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, with all the λ_j's distinct, and $[\Lambda, B] = 0$, then the matrix B is a diagonal matrix. This is because $[\Lambda, B] = 0$ implies $\Lambda B = B\Lambda$. Denoting the jth row and kth column element of B by b_{jk}, this relation yields $\lambda_j b_{jk} = \lambda_k b_{jk}$ or $(\lambda_j - \lambda_k) b_{jk} = 0$. Since $\lambda_j \neq \lambda_k$ when $j \neq k$, we find that $b_{jk} = 0$ when $j \neq k$. Hence, the matrix B is diagonal.

Lemma 2. *The ten commutation conditions in (23)–(25) imply the pairwise commutation of the symmetric matrices*

$$S,\ K,\ G^2,\ GN,\ GKG,\ \text{and}\ GSG \tag{37}$$

Proof. We begin by noting that $[G, N] = 0$ if and only if GN is a symmetric matrix (see [6], Lemma 2). That the matrices K and S commute with each of the remaining matrices listed in (37), follows directly from (23)–(25). Using the algebra of commutators, we show that the remaining matrices in this list also commute pairwise. Noting that $[G, N] = 0$, we find that

$$[G, GN] = G[G, N] + [G, G]N = 0$$

and therefore $[G^2, GN] = 0$ (see Remark 4, part 4). Also, $[GN, G] = 0$ by Remark 4, part 2. We next find that

$$\begin{aligned}[G^2, GKG] &= [G^2, G]KG + G[G^2, KG] \\ &= G[G^2, KG] = G[G^2, K]G + GK[G^2, G] \\ &= G[G^2, K]G = 0\end{aligned}$$

The second and fourth equalities follow from Remark 4, part 1, and the last equality follows from (25). Using (25) and replacing all occurrences of K by S above, we similarly get $[G^2, GSG] = 0$.

Furthermore, recalling that $[GN, G] = 0$, we see that $[GN, KG] = [GN, K]G + K[GN, G] = [GN, K]G = 0$; the last equality follows from (23). Using this result, we now find that

$$[GN, GKG] = [GN, G]KG + G[GN, KG] = 0$$

Similarly, using (23) again, we get $[GN, SG] = [GN, S]G + S[GN, G] = [GN, S]G = 0$, so that

$$[GN, GSG] = [GN, G]SG + G[GN, SG] = 0$$

Lastly, we show that the matrices GKG and GSG commute as follows:

$$\begin{aligned}(GKG)(GSG) &= GKG^2SG = GG^2KSG = GG^2SKG = GSG^2KG \\ &= (GSG)(GKG)\end{aligned}$$

In the second and fourth equalities above we have used (25), and in the third equality we have used (23). ■

Lemma 3. *Assume that* $0 < Rank(G) = 2m \leq n$ $(G \neq 0)$ *and that there exists an orthogonal matrix* $\hat{Q}$ *such that*

$$\text{(a)}\quad \Gamma = \hat{Q}^T G \hat{Q} = \text{diag}(\beta_1 J_2, \beta_2 J_2, \ldots, \beta_m J_2, 0_{n-2m}) \tag{38}$$

where $\beta_j > 0$, $j = 1, \ldots, m$, $J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, 0_{n-2m} *is the* $(n-2m)$ *by* $(n-2m)$ *zero matrix, and* $J_2^2 = -I_2$, I_2 *being the 2 by 2 identity matrix.*

$$\text{(b)}\quad \bar{N} = \hat{Q}^T N \hat{Q} = \text{diag}(\nu_1 J_2, \nu_2 J_2, \ldots, \nu_m J_2, \hat{N}_{n-2m}) \tag{39}$$

where ν_j, $j = 1, \ldots, m$, *are real numbers, and* $\hat{N}_{n-2m}$ *is an* $(n-2m)$ *by* $(n-2m)$ *skew-symmetric matrix.*

$$\text{(c)}\quad \bar{\Lambda} = \hat{Q}^T K \hat{Q} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2m}, \hat{K}_{n-2m}) \tag{40}$$

where λ_j, $j = 1, \ldots, m$, *are real numbers, and* $\hat{K}_{n-2m}$ *is an* $(n-2m)$ *by* $(n-2m)$ *symmetric matrix.*

$$\text{(d)}\quad \bar{\Sigma} = \hat{Q}^T S \hat{Q} = \text{diag}(\sigma_1, \sigma_2, \ldots, \hat{S}_{n-2m}) \tag{41}$$

where σ_j, $j = 1, \ldots, m$, *are real numbers, and* $\hat{S}_{n-2m}$ *is an* $(n-2m)$ *by* $(n-2m)$ *symmetric matrix. The numbers* ν_j, λ_j, *and* σ_j *could be zero. We shall assume throughout this article that* $\beta_j > 0$, $j = 1, \ldots m$, *something that can always be achieved by suitable permutation.*

Then the matrices Γ^2, $\Gamma\bar{N}$, $\Gamma\bar{\Lambda}\Gamma$, *and* $\Gamma\bar{\Sigma}\Gamma$ *are diagonal.*

Proof. From (38) to (41), we obtain

$$\begin{aligned}\Gamma^2 &= \hat{Q}^T G \hat{Q} \hat{Q}^T G \hat{Q} = \hat{Q}^T G^2 \hat{Q} \\ &= -\text{diag}(\beta_1^2 I_2, \beta_2^2 I_2, \ldots, \beta_m^2 I_2, 0_{n-2m})\end{aligned} \tag{42}$$

$$\begin{aligned}\Gamma\bar{N} &= \hat{Q}^T G \hat{Q} \hat{Q}^T N \hat{Q} = \hat{Q}^T (GN) \hat{Q} \\ &= -\text{diag}(\beta_1 \nu_1 I_2, \beta_2 \nu_2 I_2, \ldots, \beta_m \nu_m I_2, 0_{n-2m})\end{aligned} \tag{43}$$

$$\begin{aligned}\Gamma\bar{\Lambda}\Gamma &= \hat{Q}^T (GKG) \hat{Q} \\ &= -\text{diag}(\beta_1^2 \lambda_2, \beta_1^2 \lambda_1, \ldots, \beta_m^2 \lambda_{2m}, \beta_m^2 \lambda_{2m-1}, 0_{n-2m})\end{aligned} \tag{44}$$

and

$$\begin{aligned}\Gamma\bar{\Sigma}\Gamma &= \hat{Q}^T (GSG) \hat{Q} \\ &= -\text{diag}(\beta_1^2 \sigma_2, \beta_1^2 \sigma_1, \ldots, \beta_m^2 \sigma_{2m}, \beta_m^2 \sigma_{2m-1}, 0_{n-2m})\end{aligned} \tag{45}$$

Each of the matrices in (42)–(45) are diagonal and commute pairwise (see Remark 4, part 3). ■

We are now ready to prove the following preliminary result.

Lemma 4. *Assume that* $0 < Rank(G) = 2m \leq n$ $(G \neq 0)$. *If the ten commutation conditions given in (23)–(25) are satisfied by the* n *by* n *matrices* $G \neq 0$ N, S, *and* K, *then there exist two real orthogonal unit vectors* q_1 *and* q_2 *such that the following relations are satisfied:*

$$Gq_1 = -\beta_1 q_2, \qquad Gq_2 = \beta_1 q_1,\ \beta_1 > 0 \tag{46}$$

$$Nq_1 = -\nu_1 q_2, \qquad Nq_2 = \nu_1 q_1 \tag{47}$$

$$Kq_1 = \lambda_1 q_1, \qquad Kq_2 = \lambda_2 q_2 \tag{48}$$

$$Sq_1 = \sigma_1 q_1 \qquad Sq_2 = \sigma_2 q_2 \tag{49}$$

where ν_1, λ_1, λ_2, σ_1, *and* σ_2 *are real numbers.*

Proof. Since the conditions (23)–(25) are satisfied, by Lemma 2, the symmetric matrices S, K, G^2, GN, GKG, and GSG pairwise commute. Then according to a well-known result (see, for example, Ref. [8]), these matrices have n common linearly independent real eigenvectors. Let $\sigma(G) = (\pm i\beta_1, \ldots, \pm i\beta_m, 0, \ldots, 0)$, $\beta_j > 0$, $j = 1, 2, \ldots, m$, be the spectrum (denoted by $\sigma(\square)$) of the skew-symmetric matrix G. Then $\sigma(G^2) = (-\beta_1^2, -\beta_1^2, \ldots, -\beta_m^2, -\beta_m^2, 0, \ldots, 0)$. With no loss of generality, let q_1 be a (real) common unit eigenvector of the pairwise commuting matrices such that

$$G^2 q_1 = -\beta_1^2 q_1, \quad Kq_1 = \lambda_1 q_1, \qquad Sq_1 = \sigma_1 q_1 \tag{50}$$

$$GNq_1 = \alpha_1 q_1, \quad GKGq_1 = \mu_1 q_1, \quad GSGq_1 = \eta_1 q_1 \tag{51}$$

where $\beta_1 > 0$ is a real number, and λ_1, σ_1, α_1, μ_1, and η_1, are real numbers (some of which could be zero).

Multiplying each of the relations in (51) by G from the left, we get

$$G^2Nq_1 = \alpha_1 Gq_1,\ G^2KGq_1 = \mu_1 Gq_1,\ G^2SGq_1 = \eta_1 Gq_1$$

Noting that $[G^2, N] = [K, G^2] = [S, G^2] = 0$ (i.e., the respective pairs commute) in which the first relation follows from $[G, N] = 0$ (Remark 4, part 4), we obtain

$$N(G^2q_1) = \alpha_1 Gq_1,\ KG(G^2q_1) = \mu_1 Gq_1,\ SG(G^2q_1) = \eta_1 Gq_1 \quad (52)$$

Using the first relation in (50), (52) becomes

$$Nq_1 = -\alpha_1\beta_1^{-2}(Gq_1),\ K(Gq_1) = -\mu_1\beta_1^{-2}(Gq_1),$$
$$S(Gq_1) = -\eta_1\beta_1^{-2}(Gq_1) \quad (53)$$

From the last two relations above, we infer that $-(Gq_1)$ is a real eigenvector of both K and S; the negative sign before (Gq_1) is chosen for convenience, as we will see later. The length of this vector is given by

$$\|Gq_1\| = \sqrt{q_1^TG^TGq_1} = \sqrt{-q_1(G^2q_1)} = \sqrt{\beta_1^2q_1^Tq_1} = \beta_1 > 0$$

In the second equality above, we have used the fact that G is skew-symmetric ($G^T = -G$), and in the third equality we have used the first relation in (50). Dividing the eigenvector $-(Gq_1)$ by its length, we get

$$q_2 = \frac{-(Gq_1)}{\beta_1} \quad (54)$$

which is then a (real) common unit eigenvector of K and S. Moreover, the unit vectors q_1 and q_2 are orthogonal. This is because $q_1^Tq_2 = -q_1^TGq_1/\beta_1 = 0$, since G is skew-symmetric and the numerator is zero. From (54), we note that $Gq_1 = -\beta_1 q_2$.

The relations in (53) can now be rewritten as

$$Nq_1 = -\alpha_1\beta_1^{-2}(Gq_1) = \alpha_1\beta_1^{-1}q_2 = -\nu_1 q_2,\ Kq_2 = \lambda_2 q_2,\ Sq_2 = \sigma_2 q_2 \quad (55)$$

where we have denoted $\nu_1 := -\alpha_1\beta_1^{-1}$, $\lambda_2 := -\mu_1\beta_1^{-2}$, and $\sigma_2 := -\eta_1\beta_1^{-2}$. Furthermore, multiplication of (54) by N from the left gives

$$Nq_2 = \frac{-NGq_1}{\beta_1} = \frac{-GNq_1}{\beta_1} = -\alpha_1\beta_1^{-1}q_1 = \nu_1 q_1 \quad (56)$$

The second equality follows since G and N commute (see the first relation in (23)), and the third follows from the first relation in (51).

The first relation in (55) along with (56) then gives

$$Nq_1 = -\nu_1 q_2,\ Nq_2 = \nu_1 q_1 \quad (57)$$

and the last two relations in (50) and the last two in (55) can be summarized as

$$Kq_1 = \lambda_1 q_1,\ Kq_2 = \lambda_2 q_2 \text{ and } Sq_1 = \sigma_1 q_1,\ Sq_2 = \sigma_2 q_2 \quad (58)$$

Also, upon multiplication by G on the left, relation (54), gives

$$Gq_2 = \frac{-G^2q_1}{\beta_1} = \frac{\beta_1^2 q_1}{\beta_1} = \beta_1 q_1 \quad (59)$$

where we have used the first relation in (50) to get the second equality. Equations (54) and (59) can be summarized as

$$Gq_1 = -\beta_1 q_2,\ Gq_2 = \beta_1 q_1 \quad (60)$$

Equations (60), (57), and (58) prove the lemma. ∎

THEOREM 5. *If the ten commutation conditions given in (23)–(25) are satisfied, a real orthogonal matrix $\hat{Q}$ exists such that the n by n matrices $G \neq 0$, N, S, and K can be simultaneously transformed by an orthogonal congruence to yield the forms given in (38)–(41).*

Proof. Since the commutation relations (23)–(25) are satisfied, by Lemma 2 the matrices listed in (37) commute pairwise. We need to show that an orthogonal matrix $\hat{Q}$ exists such that $\hat{Q}^TG\hat{Q} = \Gamma$, $\hat{Q}^TN\hat{Q} = \bar{\mathrm{N}}$, $\hat{Q}^TK\hat{Q} = \bar{\Lambda}$, and $\hat{Q}^TS\hat{Q} = \bar{\Sigma}$ where Γ, $\bar{\mathrm{N}}$, $\bar{\Lambda}$, and $\bar{\Sigma}$ are as in (38)–(41).

Lemma 4 shows that when $G \neq 0$ and conditions (23)–(25) are satisfied, then we can find two orthogonal unit vectors q_1 and q_2 such that the relations (46)–(49) are satisfied. We now construct an n by n orthogonal matrix using these two unit vectors q_1 and q_2 found in this lemma as the first two columns of an orthogonal matrix

$$Q_1 = [q_1, q_2, q_3, \ldots, q_n] \quad (61)$$

the remaining orthogonal unit vectors $\{q_3, \ldots, q_n\}$ being chosen arbitrarily so that the relation $Q_1^TQ_1 = I_n$ is satisfied.

We now look at the structure of the four matrices $Q_1^TGQ_1$, $Q_1^TNQ_1$, $Q_1^TKQ_1$, and $Q_1^TSQ_1$, focusing on the first two rows (columns) of these matrices. Using (46) in Lemma 4, recalling that $G^T = -G$, and denoting δ_{jk} by the Kronecker delta, we have, for $k = 1, 2, \ldots, n$,

$$q_1^TGq_k = -q_k^TGq_1 = \beta_1 q_k^Tq_2 = \beta_1\delta_{2k} \text{ and}$$
$$q_2^TGq_k = -q_k^TGq_2 = -\beta_1 q_k^Tq_1 = -\beta_1\delta_{1k} \quad (62)$$

The relations in (62) give the elements of the first two rows (columns) of the matrix $Q_1^TGQ_1$. Similarly, using (47), for $k = 1, 2, \ldots, n$, we get the elements of the first two rows (columns) of $Q_1^TNQ_1$ as

$$q_1^TNq_k = -q_k^TNq_1 = \nu_1 q_k^Tq_2 = \nu_1\delta_{2k} \text{ and}$$
$$q_2^TNq_k = -q_k^TNq_2 = -\nu_1 q_k^Tq_1 = -\nu_1\delta_{1k} \quad (63)$$

Using (48) and (49), for $k = 1, 2, \ldots, n$, the elements of the first two rows (columns) of $Q_1^TKQ_1$ and $Q_1^TSQ_1$ are given, respectively, by

$$q_1^TKq_k = q_k^TKq_1 = \lambda_1 q_k^Tq_1 = \lambda_1\delta_{1k} \text{ and}$$
$$q_2^TKq_k = q_k^TKq_2 = \lambda_2 q_k^Tq_2 = \lambda_2\delta_{2k} \quad (64)$$

and

$$q_1^TSq_k = q_k^TSq_1 = \sigma_1 q_k^Tq_1 = \sigma_1\delta_{1k} \text{ and}$$
$$q_2^TSq_k = q_k^TSq_2 = \sigma_2 q_k^Tq_2 = \sigma_2\delta_{2k} \quad (65)$$

From (62) to (65), the structures of the four matrices, $Q_1^TGQ_1$, $Q_1^TNQ_1$, $Q_1^TKQ_1$, and $Q_1^TSQ_1$, are thus found to be as follows:

$$Q_1^TGQ_1 = \left[\begin{array}{cc|c} 0 & \beta_1 & 0 \\ -\beta_1 & 0 & \\ \hline 0 & & \hat{G}_{n-2} \end{array}\right], \qquad Q_1^TNQ_1 = \left[\begin{array}{cc|c} 0 & \nu_1 & 0 \\ -\nu_1 & 0 & \\ \hline 0 & & \hat{N}_{n-2} \end{array}\right]$$

and

$$Q_1^TKQ_1 = \left[\begin{array}{cc|c} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \\ \hline 0 & & \hat{K}_{n-2} \end{array}\right], \text{ and } \quad Q_1^TSQ_1 = \left[\begin{array}{cc|c} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & \\ \hline 0 & & \hat{S}_{n-2} \end{array}\right].$$

Since the $(n-2)$ by $(n-2)$ matrices, $\hat{G}_{n-2}$, $\hat{N}_{n-2}$, $\hat{K}_{n-2}$, and $\hat{S}_{n-2}$, satisfy the same conditions as G, N, K, and S, this procedure continues in the same manner, and after m steps we conclude that there exists an orthogonal matrix $\hat{Q}$ such that

$$\Gamma = \hat{Q}^TG\hat{Q} = \mathrm{diag}\left(\beta_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0_{n-2m}\right) \quad (66)$$

$$\bar{\mathrm{N}} = \hat{Q}^TN\hat{Q} = \mathrm{diag}\left(\nu_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \nu_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \hat{N}_{n-2m}\right) \quad (67)$$

$$\bar{\Lambda} = \hat{Q}^TK\hat{Q} = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_{2m}, \hat{K}_{n-2m}) \quad (68)$$

and

$$\bar{\Sigma} = \hat{Q}^T S \hat{Q} = \mathrm{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_{2m}, \hat{S}_{n-2m}) \tag{69}$$

which are the same as (38)–(41), in which we recall that 0_{n-2m}, $\hat{N}_{n-2m}$, $\hat{K}_{n-2m}$, and $\hat{S}_{n-2m}$ are each $(n-2m)$ by $(n-2m)$ matrices, $\hat{N}_{n-2m}$ is skew-symmetric matrix, and $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$ are symmetric matrices. Since K and S commute, the matrices $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$ in (68) and (69) commute with each other. ∎

Remark 5. The proof of Theorem 5 is constructive in that it gives an explicit method for the construction of a real orthogonal matrix $\hat{Q}$ to obtain the relations in (66)–(69). The column vectors q_1 and q_2 are found as in Lemma 4, and the unit orthonormal vectors $\{q_3, \ldots, q_n\}$ in (61), for example, can be found computationally using a robust Gram-Schmidt orthogonalization algorithm or a QR algorithm.

Remark 6. It is clear from Lemma 4 and Theorem 5 that the roles of the matrices G and N can be interchanged, with $0 < Rank\,(N) = 2m \le n$ $(N \neq 0)$. Also, the roles of K and S can be interchanged.

We next prove a lemma that will be used later.

Lemma 5. *Let K and S be (any) two 2 by 2 symmetric matrices that commute with one another, i.e., $[K, S] = 0$. Then, the ten conditions given in (23)–(25) are satisfied for any two arbitrary 2 by 2 skew-symmetric matrices G and N.*

Proof. Consider any 2 by 2 skew-symmetric matrices $G = \beta J_2$ and $N = \nu J_2$, and any orthogonal 2 by 2 matrix

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} := [q_1 q_2]$$

whose determinant, Δ, is 1. Noting that $q_1 q_2^T - q_2 q_1^T = \Delta J_2 = J_2$, we then have

$$QGQ^T = \beta Q J_2 Q^T = \beta [q_1 q_2] J_2 \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} = \beta(q_1 q_2^T - q_2 q_1^T) = \beta J_2 = G \tag{70}$$

From the first and last equality, we observe that $G = \beta Q J_2 Q^T$; similarly,

$$N = QNQ^T \text{ and } N = \nu Q J_2 Q^T \tag{71}$$

so that $GN = \beta\nu Q J_2^2 Q^T = -\beta\nu I$. Furthermore, $[G, N] = \beta\nu Q [J_2^2 - J_2^2] Q^T = 0$.

Since K and S commute, there is a real 2 by 2 orthogonal matrix Q such that $K = Q\Lambda Q^T$ and $S = Q\Sigma Q^T$ where $\Lambda = \mathrm{diag}\,(\lambda_1, \lambda_2)$ and $\Sigma = \mathrm{diag}\,(\sigma_1, \sigma_2)$ [8]. We order the two columns of this matrix Q so that its determinant is unity, and we now choose this orthogonal matrix Q, which simultaneously diagonalizes K and S, in (70) and (71). Noting that $J_2^2 = -I_2$, we then find that

$$\begin{aligned} &K = Q\Lambda Q^T,\ S = Q\Sigma Q^T,\ N = QNQ^T \\ &G^2 = \beta^2 Q J_2 J_2 Q^T = -\beta^2 Q I_2 Q^T, \\ &GN = \beta\nu Q J_2 J_2 Q^T = -\beta\nu Q I_2 Q^T = NG, \\ &GKG = \beta^2 Q J_2 Q^T Q \Lambda Q^T Q J_2 Q^T = \beta^2 Q J_2 \Lambda J_2 Q^T \\ &\quad = Q \begin{bmatrix} -\beta^2 \lambda_2 & 0 \\ 0 & -\beta^2 \lambda_1 \end{bmatrix} Q^T \end{aligned} \tag{72}$$

and similarly

$$GSG = \beta^2 Q J_2 \Sigma J_2 Q^T = Q \begin{bmatrix} -\beta^2 \sigma_2 & 0 \\ 0 & -\beta^2 \sigma_1 \end{bmatrix} Q^T$$

Using Remark 4, part 3, the conditions in (23)–(25) are automatically satisfied. ∎

We are now ready to extend Theorem 5, while still keeping the same assumptions made in it, by considering a further decomposition, when $G \neq 0$, of the matrices $\hat{N}_{n-2m}$, $\hat{K}_{n-2m}$, and $\hat{S}_{n-2m}$ shown in (67)–(69) into quasi-diagonal forms when conditions (23)–(25) are satisfied. We first consider the simpler situation when $n \le 2m+2$, i.e., the dimension, n, of each of the n by n matrices $G \neq 0$, N, S, and K does not exceed the rank, $2m$, of G by more than 2.

Consider first the case when $n = 2m$. Then the matrices $\hat{N}_{n-2m}$, $\hat{K}_{n-2m}$, and $\hat{S}_{n-2m}$ disappear from (67) to (69), and we see that the real orthogonal matrix $\hat{Q}$ obtained in Theorem 5 then simultaneously quasi-diagonalizes the four matrices G, N, S, and K, as in the statement of Theorem 2, using in it $Q = \hat{Q}$ that was obtained in Theorem 5.

Next, consider the case when $n = 2m+2$ and the commutation conditions in (23)–(25) are satisfied. Then the matrices $\hat{N}_{n-2m}$, $\hat{K}_{n-2m}$, and $\hat{S}_{n-2m}$ in (67)–(69) are each 2 by 2 matrices. Note that $\hat{N}_{n-2m}$ is skew-symmetric since N is skew-symmetric. Also, the symmetric 2 by 2 matrices, $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$, commute (because K and S commute), therefore there exists a real orthogonal 2 by 2 matrix, $\bar{Q}$, with determinant 1, such that $\bar{Q}^T \hat{K}_{n-2m} \bar{Q} = \mathrm{diag}\,(\lambda_{2m+1}, \lambda_{2m+2})$, $\bar{Q}^T \hat{S}_{n-2m} \bar{Q} = \mathrm{diag}\,(\sigma_{2m+1}, \sigma_{2m+2})$, and $\bar{Q}^T \hat{N}_{n-2m} \bar{Q} = \hat{N}_{n-2m}$ (see (72), Lemma 5). Thus, when $n = 2m+2$, having obtained the n by n orthogonal matrix $\hat{Q}$ as in Theorem 5, the real orthogonal matrix

$$Q = \hat{Q} \begin{bmatrix} I_{2m} & 0 \\ 0 & \bar{Q} \end{bmatrix} = \hat{Q}\mathrm{diag}\,(I_{2m}, \bar{Q}) \tag{73}$$

yields

$$\begin{aligned} &Q^T G Q = \mathrm{diag}\,(\beta_1 J_2, \ldots, \beta_m J_2, 0, 0), \\ &Q^T N Q = \mathrm{diag}\,(\nu_1 J_2, \ldots, \nu_m J_2, \hat{N}_{n-2m}) \end{aligned}$$

and

$$\begin{aligned} &Q^T K Q = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_{2m}, \lambda_{2m+1}, \lambda_{2m+2}), \\ &Q^T S Q = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_{2m}, \sigma_{2m+1}, \sigma_{2m+2}) \end{aligned}$$

as in the statement of Theorem 2, for this case. Note that the last two diagonal elements in $Q^T G Q$ are zero because $\mathrm{Rank}\,(G) = 2m$, and the last 2 by 2 diagonal block of $Q^T N Q$ above is the 2 by 2 matrix $\hat{N}_{n-2m}$, which could also be written as $\nu_{m+1} J_2$, as in (20).

Lastly, when $n = 2m+1$ and the square matrices $G \neq 0$, N, K, and S are each of dimension $(2m+1)$ by $(2m+1)$, the matrix $\hat{N}_{n-2m}$ in (67) is now a scalar, which must be zero, since $\hat{N}_{n-2m}$ is skew-symmetric. Similarly, $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$ in (68) and (69), respectively, are scalars, which we may denote by λ_{2m+1} and σ_{2m+1}, respectively. Then having obtained $\hat{Q}$ as in Theorem 5, with $\bar{Q}$ equal to the scalar 1 in (73), the real orthogonal matrix

$$Q = \hat{Q} \begin{bmatrix} I_{2m} & 0 \\ 0 & 1 \end{bmatrix} = \hat{Q}$$

yields

$$Q^T G Q = \mathrm{diag}\,(\beta_1 J_2, \ldots, \beta_m J_2, 0),\ Q^T N Q = \mathrm{diag}\,(\nu_1 J_2, \ldots, \nu_m J_2, 0)$$

and

$$Q^T K Q = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_{2m}, \lambda_{2m+1}),\ Q^T S Q = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_{2m}, \sigma_{2m+1})$$

as stated in Theorem 2, for this case. The three cases considered above are summarized in (19)–(22).

This analysis then leads to the following result.

THEOREM 6. *If the commutation conditions (23)–(25) in Theorem 2 are satisfied and $n \leq 2m+2$, then a real orthogonal matrix Q exists such that the matrices $G \neq 0$, N, K, and S can be simultaneously quasi-diagonalized. The resulting quasi-diagonal forms are as in (19)–(22).*

When $n = 2m$ or when $n = 2m+1$, the orthogonal matrix Q in Theorem 2 is $Q = \hat{Q}$; when $n = 2m+2$, $Q = \hat{Q}\mathrm{diag}\,(I_{2m}, \bar{Q})$. The orthogonal matrix $\hat{Q}$ is as constructed in Theorem 5, and $\bar{Q}$ is the orthogonal matrix, with determinant 1, that simultaneously diagonalizes the two 2 by 2 matrices $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$ in (68) and (69).

■

The converse of this theorem will next be proved.

THEOREM 7. *If a real orthogonal matrix Q exists such that the matrices $G \neq 0$, N, K, and S can be simultaneously orthogonally quasi-diagonalized and $n \leq 2m+2$, then the commutation conditions in (23)–(25) are satisfied.*

Proof. If an orthogonal Q exists such that (19)–(22) are true, then each of the matrices K, S, G^2, GN, GKG, GSG can be written as QXQ^T, where X is a diagonal matrix, and therefore according to Remark 4, part 3, the ten commutation conditions in (23)–(25) are satisfied. For example,

$$K = Q\mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)Q^T,\; S = Q\mathrm{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_n)Q^T$$

$$G^2 = Q\Gamma Q^T Q\Gamma Q^T = Q\Gamma^2 Q^T = -Q\mathrm{diag}\,(\beta_1^2 I_2, \ldots, \beta_m^2 I_2, 0_{n-2m})Q^T$$

and

$$GKG = -Q\mathrm{diag}\,(\beta_1^2\lambda_2, \beta_1^2\lambda_1, \ldots, \beta_m^2\lambda_{2m}, \beta_m^2\lambda_{2m-1}, 0_{n-2m})Q^T$$

Hence (by Remark 4, part 3), $[K, G^2] = [S, G^2] = [K, GKG] = [S, GKG] = 0$. The other commutation conditions can be proved similarly. ■

Theorems 6 and 7 taken together conclude the proof of Theorem 2, since we have shown that when $n \leq 2m+2$, a real orthogonal matrix Q exists such that the matrices $G \neq 0$, N, K, and S can be simultaneously quasi-diagonalized if and only if the commutation conditions (23)–(25) are satisfied. Theorem 4 follows by simply interchanging the roles of G and N in Theorem 2.

We next consider the more general case when $n > 2m+2$ and thereby prove Theorem 1. Going back to (67)–(69) in the proof of Theorem 5 and the commutation conditions (23)–(25) stated in it, we know that the square matrices $\hat{N}_{n-2m}$, $\hat{K}_{n-2m}$, and $\hat{S}_{n-2m}$ given in (67)–(69) are now each r by r matrices, with $r = n - 2m > 2$. Also, $\hat{N}_{n-2m}$ is skew-symmetric, since N is skew-symmetric; $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$ are symmetric, since K and S are symmetric; and, the matrices $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$ commute, since $[K, S] = 0$ (see (23)).

Consider first the case when $\hat{N}_{n-2m}$ is a nonzero matrix. Reference [5] proves that the three $(n-2m)$ by $(n-2m)$ matrices $\hat{N}_{n-2m}$, $\hat{K}_{n-2m}$, and $\hat{S}_{n-2m}$ can be simultaneously quasi-diagonalized by a real orthogonal matrix $\widehat{Q}_{n-2m}$ if and only if the following set of six conditions are satisfied:

$$[\hat{K}_{n-2m}, \hat{N}_{n-2m}\hat{K}_{n-2m}\hat{N}_{n-2m}] = 0, \quad [\hat{K}_{n-2m}, \hat{N}_{n-2m}\hat{S}_{n-2m}\hat{N}_{n-2m}] = 0,$$
$$[\hat{S}_{n-2m}, \hat{N}_{n-2m}\hat{K}_{n-2m}\hat{N}_{n-2m}] = 0, \quad [\hat{S}_{n-2m}, \hat{N}_{n-2m}\hat{S}_{n-2m}\hat{N}_{n-2m}] = 0,$$
$$[\hat{K}_{n-2m}, \hat{N}_{n-2m}^2] = 0, \quad \text{and} \quad [\hat{S}_{n-2m}, \hat{N}_{n-2m}^2] = 0 \tag{74}$$

and explicit methods for constructing the (real and orthogonal) matrices $\hat{Q}$ and $\widehat{Q}_{n-2m}$ are given in Theorem 5 and Ref. [5], respectively.

Next, consider the case when $\hat{N}_{n-2m} = 0$ (i.e., $\hat{N}_{n-2m}$ is a zero matrix). Now we have only the two symmetric, commuting matrices $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$ in (67)–(69) that need to be simultaneously diagonalized. It is well-known [8] that a real orthogonal matrix $\breve{Q}_{n-2m}$ exists (and there is a well-established way to obtain $\breve{Q}_{n-2m}$) such that $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$ can be simultaneously diagonalized if and only if $[\hat{K}_{n-2m}, \hat{S}_{n-2m}] = 0$. Thus, unlike the previous case, no additional commutation conditions are required because the n&s condition for the simultaneous diagonalization of $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$ is a condition that is already included in the conditions given in (23), namely, $[K, S] = 0$ [8]. It should, however, be noted that when $\hat{N}_{n-2m} = 0$ is formally substituted in the commutation conditions (74), these conditions are all automatically satisfied, indicating that no further commutation conditions are required beyond (23)–(25), i.e., those given in Theorem 5.

Hence, whether or not $\hat{N}_{n-2m}$ is zero, a real orthogonal matrix, which we denote by $\tilde{Q}_{n-2m}$, exists and can be explicitly constructed, such that the three matrices $\hat{N}_{n-2m}$, $\hat{K}_{n-2m}$, and $\hat{S}_{n-2m}$ can be simultaneously quasi-diagonalized when the six commutation conditions (74) are satisfied. ■

This brings us the following lemma.

LEMMA 6

(a) *If the commutation conditions (23)–(25) and (74) are satisfied, $G \neq 0$, and $\hat{N}_{n-2m}$ is a nonzero matrix, then there exists a real orthogonal matrix $\tilde{Q}_{n-2m} := \widehat{Q}_{n-2m}$ such that the matrices $\hat{N}_{n-2m}$, $\hat{K}_{n-2m}$, and $\hat{S}_{n-2m}$ given in (67)–(69) can be simultaneously quasi-diagonalized. The explicit procedure for finding $\tilde{Q}_{n-2m}$ is given in Ref. [5].*

(b) *If the commutation conditions (23)–(25) and (74) are satisfied, $G \neq 0$, and $\hat{N}_{n-2m} = 0$ (the zero matrix), a real orthogonal matrix $\tilde{Q}_{n-2m} := \breve{Q}_{n-2m}$ exists such that the matrices $\hat{K}_{n-2m}$ and $\hat{S}_{n-2m}$ are simultaneously diagonalized. In this case, the conditions in (74) are automatically satisfied.*

Thus, in each case, the matrix $\tilde{Q}_{n-2m}$ can be explicitly constructed.

In brief, for $G \neq 0$, when (23)–(25) and (74) are satisfied, there exists an orthogonal matrix $\tilde{Q}_{n-2m}$ that simultaneously quasi-diagonalizes $\hat{N}_{n-2m}$, $\hat{K}_{n-2m}$, and $\hat{S}_{n-2m}$ (given in (67)–(69)).

We now require the following lemma that will help us provide a set of commutation conditions equivalent to those given by the set in (74). The lemma can be viewed as a generalization of Remark 4, part 3.

LEMMA 7. *Let B and C be two n by n matrices. If there exists a real n by n orthogonal matrix Q such that $B = Q\mathrm{diag}\,(\mathrm{H}_1, \hat{B}_{n-k})Q^T$ and $C = Q\mathrm{diag}\,(\mathrm{H}_2, \hat{C}_{n-k})Q^T$ where H_1 and H_2 are each k by k diagonal matrices, then $[B, C] = 0$ if and only if $[\hat{B}_{n-k}, \hat{C}_{n-k}] = 0$. Furthermore, if $k = n$, then $[B, C] = 0$.*

Proof. First, consider an n by n matrix X that satisfies the relation $X = QYQ^T$, where $Q^TQ = I$. If $Y = 0$, the product on the right-hand side of the equality is zero and hence $X = 0$. On the other hand, if $X = 0$, then $Y = 0$ since $Y = Q^TXQ$. Hence, we observe that $X = 0$ if and only if $Y = 0$.

Then, since $Q^TQ = I$, we have

$$\begin{aligned}
X := [B, C] &= Q\left[\begin{array}{c|c} \mathrm{H}_1 & 0 \\ \hline 0 & \hat{B}_{n-k} \end{array}\right]Q^TQ\left[\begin{array}{c|c} \mathrm{H}_2 & 0 \\ \hline 0 & \hat{C}_{n-k} \end{array}\right]Q^T - Q\left[\begin{array}{c|c} \mathrm{H}_2 & 0 \\ \hline 0 & \hat{C}_{n-k} \end{array}\right]Q^TQ\left[\begin{array}{c|c} \mathrm{H}_1 & 0 \\ \hline 0 & \hat{B}_{n-k} \end{array}\right]Q^T \\
&= Q\left[\begin{array}{c|c} \mathrm{H}_1 & 0 \\ \hline 0 & \hat{B}_{n-k} \end{array}\right]\left[\begin{array}{c|c} \mathrm{H}_2 & 0 \\ \hline 0 & \hat{C}_{n-k} \end{array}\right]Q^T - Q\left[\begin{array}{c|c} \mathrm{H}_2 & 0 \\ \hline 0 & \hat{C}_{n-k} \end{array}\right]\left[\begin{array}{c|c} \mathrm{H}_1 & 0 \\ \hline 0 & \hat{B}_{n-k} \end{array}\right]Q^T \\
&= Q\left\{\left[\begin{array}{c|c} \mathrm{H}_1\mathrm{H}_2 & 0 \\ \hline 0 & \hat{B}_{n-k}\hat{C}_{n-k} \end{array}\right] - \left[\begin{array}{c|c} \mathrm{H}_2\mathrm{H}_1 & 0 \\ \hline 0 & \hat{C}_{n-k}\hat{B}_{n-k} \end{array}\right]\right\}Q^T \\
&= Q\left[\begin{array}{c|c} 0_k & 0 \\ \hline 0 & \hat{B}_{n-k}\hat{C}_{n-k} - \hat{C}_{n-k}\hat{B}_{n-k} \end{array}\right]Q^T = Q\left[\begin{array}{c|c} 0_k & 0 \\ \hline 0 & [\hat{B}_{n-k}, \hat{C}_{n-k}] \end{array}\right]Q^T := QYQ^T
\end{aligned}$$

in which the fourth equality arises because H_1 and H_2 are diagonal matrices, and therefore they commute with each other. Noting our previous observation, we then conclude that $X=[B, C]=0$ if and only if $Y=0$, which occurs if and only if $[\hat{B}_{n-k}, \hat{C}_{n-k}]=0$. Hence, $[B, C]=0$ if and only if $[\hat{B}_{n-k}, \hat{C}_{n-k}]=0$.

Furthermore, when $k=n$, the blocks $\hat{B}_{n-k}$ and $\hat{C}_{n-k}$ in the matrices B and C, respectively, disappear, and $Y=0_n$, where 0_n is the n by n zero matrix, so that $X=[B, C]=0$, something we knew already from Remark 4, part 3. ■

LEMMA 8. *When the commutation conditions (23)–(25) are satisfied, the six commutation conditions in (74) are equivalent to the following set of commutation conditions:*

$$[K, NKN]=0, \quad [K, NSN]=0$$
$$[S, NKN]=0, \quad [S, NSN]=0 \tag{75}$$

and

$$[K, N^2]=0, \qquad [S, N^2]=0$$

Proof. Since the commutation conditions (23)–(25) are satisfied, by Theorem 5, there exists an orthogonal matrix $\hat{Q}$ that satisfies (67)–(69) so that

$$N=\hat{Q}\bar{\mathrm{N}}\hat{Q}^T, \quad K=\hat{Q}\bar{\Lambda}\hat{Q}^T, \quad \text{and } S=\hat{Q}\bar{\Sigma}\hat{Q}^T$$

Noting the forms of $\bar{\mathrm{N}}$, $\bar{\Lambda}$, and $\bar{\Sigma}$ in (67)–(69), we then have

$$K=\hat{Q}\bar{\Lambda}\hat{Q}^T=\hat{Q}\mathrm{diag}\,(\lambda_1, \ldots, \lambda_{2m}, \hat{K}_{n-2m})\hat{Q}^T$$
$$S=Q\bar{\Sigma}Q^T=\hat{Q}\mathrm{diag}\,(\sigma_1, \ldots, \sigma_{2m}, \hat{S}_{n-2m})\hat{Q}^T$$
$$NKN=\hat{Q}\bar{\mathrm{N}}\bar{\Lambda}\bar{\mathrm{N}}\hat{Q}^T$$
$$=-\hat{Q}\mathrm{diag}\,(v_1^2\lambda_2, v_1^2\lambda_1, \ldots, v_m^2\lambda_{2m}, v_m^2\lambda_{2m-1}, \hat{N}_{n-2m}\hat{K}_{n-2m}\hat{N}_{n-2m})\hat{Q}^T$$
$$NSN=\hat{Q}\bar{\mathrm{N}}\bar{\Sigma}\bar{\mathrm{N}}\hat{Q}^T=$$
$$-\hat{Q}\mathrm{diag}\,(v_1^2\sigma_2, v_1^2\sigma_1, \ldots, v_m^2\sigma_{2m}, v_m^2\sigma_{2m-1}, \hat{N}_{n-2m}\hat{S}_{n-2m}\hat{N}_{n-2m})\hat{Q}^T$$

and

$$N^2=\hat{Q}\bar{\mathrm{N}}^2\hat{Q}^T=-\hat{Q}\mathrm{diag}\,(v_1^2, v_1^2, \ldots, v_m^2, v_m^2, \hat{N}_{n-2m}^2)\hat{Q}^T$$

We note that each of the matrices S, K, NKN, NSN, and N^2 has the structure $\hat{Q}\mathrm{diag}\,(\mathrm{H}, B_{n-2m})\hat{Q}^T$, where H is a diagonal $2m$ by $2m$ matrix. Using Lemma 7 with $k=2m$ in it, we see that each condition in the set (74) is satisfied if and only if the corresponding condition in the set (75) is also satisfied. For example, the condition $[\hat{K}_{n-2m}, \hat{N}_{n-2m}\hat{K}_{n-2m}\hat{N}_{n-2m}]=0$ in the set (74) is equivalent (by Lemma 7) to the corresponding condition $[K, NKN]=0$ in the set (75). Similarly, the condition $[\hat{K}_{n-2m}, \hat{N}_{n-2m}^2]=0$ in the set (74) is equivalent to the condition $[K, N^2]=0$ in (75), and $[\hat{S}_{n-2m}, \hat{N}_{n-2m}\hat{S}_{n-2m}\hat{N}_{n-2m}]=0$ is equivalent to $[S, NSN]=0$. ■

Remark 7. The ten commutation conditions (23)–(25) and the six in (75) comprise the 16 conditions given in (18) in Theorem 1. The conditions involve the 11 matrices G, N, S, K, G^2, GN, GKG, GSG, N^2, NKN, and NSN.

Observe that the commutators $[G, K]$, $[G, S]$, $[N, K]$, and $[N, S]$ are conspicuously *absent* from this set of 16 conditions.

THEOREM 8. *If the commutation conditions given in (23)–(25) and in the set (75) are satisfied, then there exists a real orthogonal matrix Q such that the n by n matrices $G\neq 0$, N, K, and S can be simultaneously quasi-diagonalized and have the form*

$$\Gamma=Q^TGQ=\mathrm{diag}\,(\beta_1J_2, \ldots, \beta_mJ_2, 0_{n-2m}) \tag{76}$$

$$\mathrm{N}=Q^TNQ=\mathrm{diag}\,(\nu_1J_2, \ldots, \nu_mJ_2, \ldots, v_{n/2}J_2)\ \textit{for n even}$$
$$=\mathrm{diag}\,(\nu_1J_2, \ldots, \nu_mJ_2, \ldots, v_{(n-1)/2}J_{(n-1)/2}, 0)\ \textit{for n odd} \tag{77}$$

$$\Lambda=Q^TKQ=\mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{78}$$

and

$$\Sigma=Q^TSQ=\mathrm{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{79}$$

where $\beta_j>0$, $j=1, \ldots, m$, and all the λ_j's, v_j's, and σ_j's are real numbers.

Proof. If the ten commutation conditions (23)–(25) are satisfied, we have shown in Theorem 5 that a real orthogonal n by n matrix $\hat{Q}$ exists such that $\Gamma=\hat{Q}^TG\hat{Q}$, $\bar{\mathrm{N}}=\hat{Q}^TN\hat{Q}$, $\bar{\Lambda}=\hat{Q}^TK\hat{Q}$, and $\bar{\Sigma}=\hat{Q}^TS\hat{Q}$ (see (67)–(69)); furthermore, the matrix $\hat{Q}$ is explicitly obtained.

By Lemma 8, the set of the commutation conditions (75) is equivalent to the set (74). Using Lemma 6, we then find that if the conditions in (75) are additionally satisfied, then the matrices $\hat{N}_{n-2m}$, $\hat{K}_{n-2m}$, and $\hat{S}_{n-2m}$ can be simultaneously quasi-diagonalized by a real orthogonal matrix $\tilde{Q}_{n-2m}$, which can be explicitly constructed. Hence, the real orthogonal matrix $Q=\hat{Q}\mathrm{diag}\,(I_{2m}, \tilde{Q}_{n-2m})$ simultaneously quasi-diagonalizes the matrices $G\neq 0$, N, K, and S. The quasi-diagonal forms are given in (76)–(79). ■

We now prove the converse of Theorem 8.

THEOREM 9. *If a real orthogonal matrix Q exists such that it simultaneously quasi-diagonalizes G, N, K, and S as in (76)–(79), then the commutation conditions given in (23)–(25) and in (75) are satisfied.*

Proof. If the relations (76)–(79) are satisfied, then with $Q^TQ=I_n$

$$G=Q\Gamma Q^T,\ N=Q\mathrm{N}Q^T,\ Q^TSQ=\Sigma, \text{ and } Q^TKQ=\Lambda$$

Hence,

$$S=Q\mathrm{diag}\,(\sigma_1, \ldots, \sigma_n)Q^T,\ K=Q\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)Q^T,$$
$$G^2=Q\Gamma Q^TQ\Gamma Q^T=Q\Gamma^2Q^T=-Q\mathrm{diag}\,(\beta_1^2I_2, \ldots, \beta_m^2I_2, 0_{n-2m})Q^T,$$
$$GN=-Q\mathrm{diag}\,(\beta_1v_1I_2 \ldots, \beta_mv_mI_2, 0_{n-2m})Q^T=NG,$$
$$GKG=-Q\mathrm{diag}\,(\beta_1^2\lambda_2, \beta_1^2\lambda_1, \ldots, \beta_m^2\lambda_{2m}, \beta_m^2\lambda_{2m-1}, 0_{n-2m})Q^T,$$
$$GSG=-Q\mathrm{diag}\,(\beta_1^2\sigma_2, \beta_1^2\sigma_1, \ldots, \beta_m^2\sigma_{2m}, \beta_m^2\sigma_{2m-1}, 0_{n-2m})Q^T$$
$$N^2=-Q\mathrm{diag}\,(\nu_1^2I_2, \ldots, \nu_{n/2}^2I_2)Q^T \text{for } n \text{ even}$$
$$=-Q\mathrm{diag}\,(\nu_1^2I_2, \ldots, \nu_{(n-1)/2}^2I_2, 0)Q^T \text{for } n \text{ odd} \tag{80}$$
$$NKN=-Q\mathrm{diag}\,(v_1^2\lambda_2, v_1^2\lambda_1, \ldots, v_{n/2}^2\lambda_n)Q^T \text{for } n \text{ even}$$
$$=-Q\mathrm{diag}\,(v_1^2\lambda_2, v_1^2\lambda_1, \ldots, v_{(n-1)/2}^2\lambda_n, 0)Q^T \text{for } n \text{ odd}$$

and

$$NSN=-Q\mathrm{diag}\,(v_1^2\sigma_2, v_1^2\sigma_1, \ldots, v_{n/2}^2\sigma_n)Q^T \text{for } n \text{ even}$$
$$=-Q\mathrm{diag}\,(v_1^2\sigma_2, v_1^2\sigma_1, \ldots, v_{(n-1)/2}^2\sigma_n, 0)Q^T \text{for } n \text{ odd}$$

All nine of the matrices in (80) are of the form QXQ^T, where X is a diagonal matrix, and the product $GN=NG$. By Lemma 7, all the 16 commutators in (23)–(25) and in (75) are therefore zero. ■

From Theorems 8 and 9, our central result in Theorem 1 is proved. Theorem 3 follows by simply interchanging the roles of G and N in it. Having proved the four central theorems, we obtain the following auxiliary results by successively setting $K=0$, $S=0$, and $N=0$ in them.

Theorem 10. *There exists a real orthogonal matrix Q such that the n by n matrices $G \neq 0$, S, and N can be simultaneously quasi-diagonalized and have the form*

$$Q^T GQ = \Gamma = \text{diag}\,(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m})$$
$$\Sigma = Q^T SQ = \text{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_n)$$

and

$$Q^T NQ = \mathrm{N} = \text{diag}\,(\nu_1 J_2, \ldots, \nu_{n/2} J_2) \quad \textit{for } n \textit{ even}$$
$$= \text{diag}\,(\nu_1 J_2, \ldots, \nu_{(n-1)/2} J_2, 0) \quad \textit{for } n \textit{ odd}$$

where $\beta_j > 0$, $j = 1, \ldots, m$, and all the v_j's and σ_j's are real numbers, if and only if the following set of conditions are satisfied:

$$[G, N] = 0, [S, GN] = 0, [S, GSG] = 0, [S, NSN] = 0$$
$$[S, G^2] = 0, [S, N^2] = 0 \tag{81}$$

Proof. Set $K=0$ in Theorem 1. The n&s conditions in (18) reduce to (81). ■

Theorem 11. *There exists a real orthogonal matrix Q such that the n by n matrices $G \neq 0$, K, and N can be simultaneously quasi-diagonalized and have the form*

$$Q^T KQ = \Lambda = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$$
$$Q^T GQ = \Gamma = \text{diag}\,(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m})$$

and

$$Q^T NQ = \mathrm{N} = \text{diag}\,(\nu_1 J_2, \ldots, \nu_{n/2} J_2) \quad \textit{for } n \textit{ even}$$
$$= \text{diag}\,(\nu_1 J_2, \ldots, \nu_{(n-1)/2} J_2, 0) \quad \textit{for } n \textit{ odd}$$

where $\beta_j > 0$, $j = 1, \ldots, m$, and all the v_j's and λ_j's are real numbers if and only if the following set of conditions are satisfied:

$$[G, N] = 0, \quad [K, GN] = 0, \quad [K, GKG] = 0, \quad [K, NKN] = 0,$$
$$[K, G^2] = 0, \quad [K, N^2] = 0 \tag{82}$$

Proof. We set the matrix $S=0$ in Theorem 1. The n&s conditions in (18) yield (82) for the simultaneous orthogonal quasi-diagonalization of the three matrices G, K, and N. ■

Remark 8. Theorem 11, which was obtained earlier in Ref. [6], is seen to follow directly from Theorem 1. The n&s conditions for the simultaneous orthogonal quasi-diagonalization of the matrices G, K, and $N \neq 0$ (i.e., with $0 < Rank\,(N) = 2m \leq n$) can be obtained using Theorem 3, and setting $S=0$ in it. The quasi-diagonal forms for the three matrices are given in (26), (28), and (27); the n&s conditions can be obtained from (18). One could also obtain these n&s conditions from (82) by simply exchanging in it G with N, which leaves the set of commutation conditions in (82) unchanged.

Theorem 12. *There exists a real orthogonal matrix Q such that the n by n matrices G, K, and S, with $0 < Rank(G) = 2m \leq n$, can be simultaneously quasi-diagonalized and have the form*

$$\Gamma = Q^T GQ = \text{diag}\,(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m}) \tag{83}$$

$$\Lambda = Q^T KQ = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{84}$$

and

$$\Sigma = Q^T SQ = \text{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{85}$$

where $\beta_j > 0$, $j = 1, \ldots, m$, and all the λ_j's and σ_j's are real numbers if and only if the following set of commutation conditions

$$[K, S] = 0, \quad [K, GKG] = 0, \quad [K, GSG] = 0, \quad [S, GKG] = 0,$$
$$[S, GSG] = 0, \quad [K, G^2] = 0, \quad [S, G^2] = 0 \tag{86}$$

are satisfied.

Proof. Set $N=0$ in Theorem 1. We find from (15) that $N=0$, and the set of commutation conditions in (18) reduce to (86). ■

Observe that Theorem 12 is obtained trivially by setting $N=0$ in Theorem 1. The theorem shows that any three n by n matrices G, K, and S (where G is skew-symmetric, while K and S are both symmetric), with $0 < Rank\,(G) = 2m \leq n$, can be simultaneously orthogonally quasi-diagonalized to yield the forms given in (83)–(85) if and only if the commutation conditions (86) are satisfied. This leads to the following result.

Theorem 13. *There exists a real orthogonal matrix Q such that the n by n matrices N, K, and S, with $0 < Rank\,(N) = 2m \leq n$, can be simultaneously quasi-diagonalized and have the form*

$$\mathrm{N} = Q^T NQ = \text{diag}\,(v_1 J_2, \ldots, v_m J_2, 0_{n-2m}) \tag{87}$$

$$\Lambda = Q^T KQ = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{88}$$

and

$$\Sigma = Q^T SQ = \text{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{89}$$

where $v_j > 0$, $j = 1, \ldots, m$, and all the λ_j's and σ_j's are real numbers if and only if the following set of commutation conditions

$$[K, S] = 0, \quad [K, NKN] = 0, \quad [K, NSN] = 0, \quad [S, NKN] = 0,$$
$$[S, NSN] = 0, \quad [K, N^2] = 0, \quad [S, N^2] = 0 \tag{90}$$

are satisfied.

Proof. In Theorem 12, exchange the roles of the skew-symmetric matrix G and the skew-symmetric matrix N. This could more directly have been obtained by setting $G=0$ in Theorem 3. ■

Remark 9. Theorems 12 and 13 give the n&s conditions for the simultaneous orthogonal quasi-diagonalization of the matrices S, $G \neq 0$, K and the simultaneous orthogonal quasi-diagonalization of the matrices S, K, $N \neq 0$, respectively. They trivially follow from Theorems 1 and 3.

Remark 10. Theorem 1 gives the n&s conditions in (18) for the simultaneous orthogonal quasi-diagonalization of the matrix quadruplet $\{S, G, K, N\}$, under the proviso that $0 < Rank\,(G) = 2m \leq n$. Under this proviso, Theorems 10, 11, and 12 give the n&s conditions for the simultaneous orthogonal quasi-diagonalization of the three matrix triplets $\{S, G, N\}$, $\{G, N, K\}$, and $\{S, G, K\}$. These n&s conditions and the corresponding quasi-diagonal forms are obtained directly from Theorem 1 by simply setting $K=0$, $S=0$, and $N=0$, respectively. To obtain the n&s

conditions for $\{S, N, K\}$, $N \neq 0$, and the corresponding quasi-diagonal forms, the n&s conditions (and the quasi-diagonal forms) for the simultaneous orthogonal quasi-diagonalization of the triplet $\{S, G, K\}$ with $G \neq 0$ (Theorem 12) can be used with an exchange of the symbols G and N (Theorem 13). As mentioned before, an alternative and simpler way for handling the matrix triplet $\{S, N, K\}$, $N \neq 0$, is to use Theorem 3, and set $G = 0$ in it.

Remark 11. One can also obtain from Theorem 1 the n&s conditions for the simultaneous orthogonal quasi-diagonalization of the doublet $\{G, K\}$, under the proviso that $G \neq 0$, by simply setting $S = 0$ and $N = 0$ in (18) (and in the quasi-diagonal forms), yielding the conditions $[K, G^2] = 0$ and $[K, GKG] = 0$, which were obtained earlier in Ref. [3]. Replacing K by S, the n&s conditions for the simultaneous orthogonal quasi-diagonalization of $\{G, S\}$ are $[S, G^2] = 0$ and $[S, GSG] = 0$. Also, replacing G by N in the first sentence of this remark, under the proviso that $N \neq 0$, the n&s conditions for the simultaneous orthogonal quasi-diagonalization of $\{N, K\}$ are $[K, N^2] = 0$ and $[K, NKN] = 0$. The latter two relations can also be obtained by formally setting $G = 0$ and $S = 0$ in (18). Likewise, the n&s conditions for the quasi-diagonalization of the doublet $\{S, N\}$, $N \neq 0$, are simply obtained by formally setting $G = 0$ and $K = 0$ in (18); they are $[S, N^2] = 0$ and $[S, NSN] = 0$. The n&s condition for the simultaneous quasi-diagonalization of $\{G, N\}$ is obtained from (18) by setting $K = 0$ and $S = 0$; it is $[G, N] = 0$, as found earlier in Ref. [6]. Lastly, the n&s condition for the two matrices K and S, which are both symmetric, to be simultaneously diagonalized is well-known [8] and is $[K, S] = 0$ which can again be obtained by formally setting $G = 0$ and $N = 0$ in (18).

3 Uncoupling of Linear MDOF Structural and Mechanical Systems

In this section, we use the theorems developed to uncouple the system described in (12), which we repeat here for convenience

$$\ddot{x} + \underbrace{(S + G)}_{D}\dot{x} + \underbrace{(K + N)}_{R}x = f(t) \tag{12}$$

and present our first result.

Result 1. The necessary and sufficient (n&s) conditions for a real orthogonal matrix Q to exist such that the MDOF dynamical system described by (12) with $0 < \mathrm{Rank}(G) = 2m \leq n$ $(G \neq 0)$ can be uncoupled, through quasi-diagonalization, by using the orthogonal coordinate transformation $x = Qp$ into independent subsystems that have at most two degrees-of-freedom are ((23)–(25) and (75)):

$$\begin{array}{llll} [G, N] = 0, & [K, S] = 0, & [K, GN] = 0, & [S, GN] = 0, \\ [K, GKG] = 0, & [K, GSG] = 0, & [K, NKN] = 0, & [K, NSN] = 0, \\ [S, GKG] = 0, & [S, GSG] = 0, & [S, NKN] = 0, & [S, NSN] = 0, \\ [K, G^2] = 0, & [S, G^2] = 0, & [K, N^2] = 0, & [S, N^2] = 0 \end{array} \tag{91}$$

The equation describing the uncoupled system in terms of the principal coordinate p is

$$\ddot{p} + (\Sigma + \Gamma)\dot{p} + (\Lambda + \mathrm{N})p = Q^T f(t) \tag{92}$$

where

$$\begin{aligned} \Sigma &= Q^T SQ = \mathrm{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_n) \\ \Gamma &= Q^T GQ = \mathrm{diag}\,(\beta_1 J_2, \ldots, \beta_m J_2, 0_{n-2m}) \\ \Lambda &= Q^T KQ = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \end{aligned} \tag{93}$$

and

$$\begin{aligned} \mathrm{N} = Q^T NQ &= \mathrm{diag}\,(\nu_1 J_2, \ldots, \nu_m J_2, \ldots, \nu_{n/2} J_2)\; \textit{for n even} \\ &= \mathrm{diag}\,(\nu_1 J_2, \ldots, \nu_m J_2, \ldots, \nu_{(n-1)/2} J_{(n-1)/2}, 0)\; \textit{for n odd} \end{aligned}$$

with $\beta_j > 0$, $j = 1, \ldots, m$, and all the ν_j's, λ_j's, and σ_j's are real numbers.

Each uncoupled, independent, real two-degree-of-freedom subsystem in (92) has the same specific (matrix) structure, namely,

$$\begin{bmatrix} \ddot{a} \\ \ddot{b} \end{bmatrix} + \begin{bmatrix} \sigma & \beta \\ -\beta & \tilde{\sigma} \end{bmatrix}\begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix} + \begin{bmatrix} \lambda & \nu \\ -\nu & \tilde{\lambda} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \tag{94}$$

Both the real coefficient matrices on the left-hand side of the equation in (94) have the same structure; in each of them, the off-diagonal elements have the same absolute value but opposite signs. On the right-hand side, $g_1(t)$ and $g_2(t)$ are real functions of time, t.

Proof. We use the coordinate change $x = Qp$ in (12) where Q is a real orthogonal matrix, and premultiply (12) by Q^T to yield

$$\ddot{p} + Q^T(S + G)Q\dot{p} + Q^T(K + N)Qp = Q^T f(t) \tag{95}$$

in which we shall refer to p as the principal coordinate. Since the necessary and sufficient commutation conditions (91) are satisfied, Theorem 1 states that a real orthogonal matrix Q exists such that the matrices S, G, K, and N, with $0 < \mathit{Rank}\,(G) = 2m \leq n$, can be simultaneously quasi-diagonalized so that (95) can be rewritten as

$$\ddot{p} + (\Sigma + \Gamma)\dot{p} + (\Lambda + \mathrm{N})p = Q^T f(t) \tag{96}$$

where Σ, Γ, Λ, and N are as in (14)–(17).

As seen from the right-hand sides of (93), the system uncouples into independent subsystems each with at most two degrees-of-freedom in the principal coordinate p, thereby expressing (95) in its canonical (simplest and maximally uncoupled) form. The structure of the matrices of each two-degree-of-freedom subsystem shown in (94) follows directly from (14) to (17). ■

Remark 12. When the n&s conditions in (91) are satisfied, *every* real uncoupled two-degree-of-freedom subsystem generated through simultaneous orthogonal quasi-diagonalization has the structure shown in (94). As mentioned before, the off-diagonal terms in each of the two coefficient matrices on the left-hand side of this equation have the same absolute value, but opposite signs.

Thus, Results 1 and 3 may be thought of as providing the n&s conditions for the uncoupling of an MDOF system into at most two-degree-of-freedom subsystems where each two-degree-of-freedom subsystem has the specific structure given in (94).

More specifically, the elements of the two real matrices in (94) can be identified as follows: σ and $\tilde{\sigma}$ are (real) eigenvalues of S; $\pm i\beta$ are (pure imaginary) eigenvalues of G; λ and $\tilde{\lambda}$ are (real) eigenvalues of K; $\pm i\nu$ are (pure imaginary) eigenvalues of N (some of these elements can be zero).

The 16 n&s conditions in (91) are indeed a large number of conditions, but as we will show in the next subsection, the number of commutation conditions can be greatly reduced in naturally occurring systems and in commonly found aerospace, civil, and mechanical engineering systems.

Example 1. Consider the six-degree-of-freedom system described by (12) with the matrices

$$S=\begin{bmatrix}0.4 & -0.1 & 0 & 0 & 0 & 0\\ -0.1 & 0.4 & -0.1 & 0 & 0 & 0\\ 0 & -0.1 & 0.5 & -0.2 & 0 & 0\\ 0 & 0 & -0.2 & 0.6 & -0.2 & 0\\ 0 & 0 & 0 & -0.2 & 0.55 & -0.15\\ 0 & 0 & 0 & 0 & -0.15 & 0.4\end{bmatrix},\ K=\begin{bmatrix}2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 3 & -2 & 0 & 0\\ 0 & 0 & -2 & 4 & -2 & 0\\ 0 & 0 & 0 & -2 & 3.5 & -1.5\\ 0 & 0 & 0 & 0 & -1.5 & 2\end{bmatrix},$$

$$G=\begin{bmatrix}0 & -0.1427 & -0.4258 & -0.5549 & -0.6217 & -0.5776\\ 0.1427 & 0 & -0.4433 & -0.6699 & -0.8109 & -0.7904\\ 0.4258 & 0.4433 & 0 & -0.2751 & -0.4878 & -0.5637\\ 0.5549 & 0.6699 & 0.2751 & 0 & -0.2341 & -0.3614\\ 0.6217 & 0.8109 & 0.4878 & 0.2341 & 0 & -0.1613\\ 0.5776 & 0.7904 & 0.5637 & 0.3614 & 0.1613 & 0\end{bmatrix},$$

$$N=\begin{bmatrix}0 & 0.0088 & -0.0591 & 0.1186 & -0.1021 & 0.0374\\ -0.0088 & 0 & 0.0941 & -0.2412 & 0.2430 & -0.0992\\ 0.0591 & -0.0941 & 0 & 0.3538 & -0.5432 & 0.2681\\ -0.1186 & 0.2412 & -0.3538 & 0 & 0.4789 & -0.3143\\ 0.1021 & -0.2430 & 0.5432 & -0.4789 & 0 & 0.1197\\ -0.0374 & 0.0992 & -0.2681 & 0.3143 & -0.1197 & 0\end{bmatrix}$$

and

$$f(t)=\begin{bmatrix}f_1(t) & f_2(t) & 0 & 0 & 0 & 0\end{bmatrix}^T$$

For brevity, numerical values are shown up to 4 decimal places. A short computation shows that the commutators in (91) are all zero, and therefore, by Result 1, there exists an orthogonal matrix Q that simultaneously quasi-diagonalizes K, S, G, and N. The ranks of the 6 by 6 matrices G and N are both 2, and their spectra are $\{\pm 2i, 0, 0, 0, 0\}$ and $\{\pm i, 0, 0, 0, 0\}$, respectively. We note that $n=6$ and $m=1$, so that $n>2m+2$.

Upon using the coordinate transformation $x=Qp$, with the orthogonal matrix

$$Q=\begin{bmatrix}0.2004 & 0.5170 & -0.5175 & -0.6287 & 0.1704 & -0.0200\\ 0.3679 & 0.5930 & -0.1183 & 0.6025 & -0.3568 & 0.0933\\ 0.4750 & 0.1632 & 0.4905 & 0.0514 & 0.5768 & -0.4146\\ 0.4896 & -0.1213 & 0.3604 & -0.3002 & -0.1371 & 0.7119\\ 0.4640 & -0.3541 & -0.0888 & -0.2078 & -0.5703 & -0.5320\\ 0.3791 & -0.4630 & -0.5830 & 0.3252 & 0.4086 & 0.1713\end{bmatrix}$$

(92) gives the following four independent subsystems in terms of the principal six-vector p as

$$\begin{bmatrix}\ddot{p}_1\\ \ddot{p}_2\end{bmatrix}+\begin{bmatrix}0.2164 & 2\\ -2 & 0.2853\end{bmatrix}\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix}+\begin{bmatrix}0.1641 & 0\\ 0 & 0.8530\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}=\begin{bmatrix}0.2004 & 0.3679\\ 0.5170 & 0.5930\end{bmatrix}\begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}$$

$$\ddot{p}_3+0.3771\dot{p}_3+1.7714p_3=-0.5175f_1(t)-0.1183f_2(t)$$

$$\ddot{p}_4+0.4958\dot{p}_4+2.9583p_4=-0.6287f_1(t)+0.6025f_2(t)\quad \text{Far1da\%}$$

and

$$\begin{bmatrix}\ddot{p}_5\\ \ddot{p}_6\end{bmatrix}+\begin{bmatrix}0.6094 & 0\\ 0 & 0.8659\end{bmatrix}\begin{bmatrix}\dot{p}_5\\ \dot{p}_6\end{bmatrix}+\begin{bmatrix}4.0939 & 1\\ -1 & 6.6593\end{bmatrix}\begin{bmatrix}p_5\\ p_6\end{bmatrix}=\begin{bmatrix}0.1704 & -0.3568\\ -0.0200 & 0.0933\end{bmatrix}\begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}$$

Thus, the MDOF system described by (12) uncouples into two subsystems each with two degrees-of-freedom and two subsystems each with a single-degree-of-freedom. Note that the form of each two-degree-of-freedom subsystem is as in (94).

Result 2. The set of commutation conditions (91) gives the necessary and sufficient (n&s) conditions for a real orthogonal matrix Q to exist such that the MDOF dynamical system described by (12), with $0<\text{Rank}\,(N)=2m\le n$ $(N\neq 0)$, uncouples, through quasi-diagonalization, into independent subsystems that have at most two degrees-of-freedom using the orthogonal coordinate transformation $x=Qp$. The equation describing the uncoupled system in terms of the principal coordinate p is

$$\ddot{p}+(\Sigma+\Gamma)\dot{p}+(\Lambda+\mathrm{N})p=Q^Tf(t) \tag{97}$$

where

$$\begin{aligned}\Sigma &= Q^TSQ=\text{diag}\,(\sigma_1,\sigma_2,\ldots,\sigma_n)\\ \Gamma &= Q^TGQ=\text{diag}\,(\beta_1J_2,\ldots,\beta_mJ_2,\ldots,\beta_{n/2}J_2)\ \text{for } n \text{ even}\\ &= \text{diag}\,(\beta_1J_2,\ldots,\beta_mJ_2,\ldots,\beta_{(n-1)/2}J_{(n-1)/2},0)\ \text{for } n \text{ odd}\\ \Lambda &= Q^TKQ=\text{diag}\,(\lambda_1,\lambda_2,\ldots,\lambda_n)\end{aligned} \tag{98}$$

and

$$\mathrm{N} = Q^T N Q = \mathrm{diag}\,(v_1 J_2, \ldots, v_m J_2, 0_{n-2m})$$

with $v_j > 0$, $j = 1, \ldots, m$, and all the β_j's, λ_j's, and σ_j's are real numbers.

Each uncoupled, independent, real two-degree-of-freedom subsystem in (97) has the specific matrix structure given in (94).

Proof. By Theorem 3, the n&s conditions for the existence of an orthogonal matrix Q, which simultaneously quasi-diagonalizes the matrix quadruplet $\{S, G, K, N\}$, are satisfied. Using the transformation $x = Qp$ as in the previous proof, (12) transforms to (97), and the quasi-diagonal forms engendered in (98) come directly from Theorem 3. From (26) to (29), each uncoupled, independent, real two-degree-of-freedom subsystem in (97) has the specific matrix structure given in (94). ∎

Remark 13. In the special case when each matrix in the quadruplet $\{S, G, K, N\}$ is a 2 by 2 matrix, the condition $[K, S] = 0$ implies that the other 15 commutation conditions in (91) are also satisfied. Direct computations prove this easily.

Remark 14. Since simultaneous orthogonal quasi-diagonalization always generates uncoupled two-degree-of-freedom subsystems each having the same specific matrix structure shown in (94) (see Remark 12 also), this poses a restriction on the quasi-diagonalization approach for uncoupling general MDOF systems. One can envision MDOF systems, which when uncoupled, have two-degree-of-freedom subsystems that do not conform to this restricted structure in (94). In such cases, the commutation conditions in (91) continue to be sufficient conditions for uncoupling MDOF systems, though, not necessary.

To illustrate Remark 14, we consider the following simple example.

Example 2. Consider a simple MDOF system composed of a three-degree-of-freedom system described by the equation

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 5 & -3 & 0 \\ -4 & 7 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{R} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad (99)$$

As seen from (99), this system decouples into two independent subsystems: the two-degree-of-freedom subsystem

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 5 & -3 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad (100)$$

and the single-degree-of-freedom subsystem

$$\ddot{x}_3 + \dot{x} + 3x = 0 \quad (101)$$

A simple computation of the symmetric parts of D and R in (99) shows that $[K, S] \neq 0$. Therefore, the n&s conditions in (91) for simultaneous orthogonal quasi-diagonalization are not satisfied. Hence, system (99) cannot be uncoupled through quasi-diagonalization; and yet, this MDOF system is clearly uncoupled into the two independent subsystems shown in (100) and (101). Note that the structure of the matrices of the two-degree-of-freedom subsystem in (100) is such that this subsystem cannot be reduced to the specific form (94) by an orthogonal transformation.

Corollary 1. *If the four matrices S, G, K, and N commute pairwise, the MDOF dynamical system described by (12) can be uncoupled using a real orthogonal coordinate transformation* $x = Qp$ *into independent subsystems that have at most two degrees-of-freedom.*

Proof. If the four matrices commute pairwise, we have a total of six commutation conditions. Using them, the n&s commutation conditions in (91) for quasi-diagonalization are all satisfied. By Results 1 and 2, a real orthogonal matrix Q exists so that (12) transforms, using the real coordinate transformation $x = Qp$, to the form given in (97). The quasi-diagonal matrices are given in (93) when $G \neq 0$, and in (98) when $N \neq 0$. If $G \neq 0$ and $N \neq 0$, either (93) or (98) can be used. ∎

Remark 15. In general, though the 16 n&s conditions given in (91) impose a larger number of conditions on the four matrices S, G, K, and N than the six pairwise commutation conditions in Corollary 1, these six conditions impose more stringent restrictions on the nature of the four matrices so that they can be simultaneously orthogonally quasi-diagonalized, leading to uncoupled independent subsystems of the linear MDOF system. For example, when $G \neq 0$ and the pairwise commutation conditions in the Corollary 1 are satisfied, there exists a real orthogonal matrix Q such that (12) transforms to (92). From the relations in (93), we find that $G = Q\Gamma Q^T$ and $K = Q\Lambda Q^T$ where the matrices Γ and Λ are quasi-diagonal and diagonal. The pairwise commutation condition $[G, K] = 0$ in Corollary 1 (see Remark 7, for comparison), requires that $Q(\Gamma\Lambda - \Lambda\Gamma)Q^T = 0$, implying $\Gamma\Lambda = \Lambda\Gamma$. Noting the quasi-diagonal structure of Γ, for this to be true the matrix Λ, which contains the eigenvalues of K along its diagonal, must have the structure

$$\Lambda = \mathrm{diag}\,(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_m, \lambda_m, \lambda_{2m+1}, \ldots, \lambda_n)$$

since all $\beta_j > 0$. This restricts the matrix K to having m pairs of multiple eigenvalues to be eligible for simultaneous quasi-diagonalization, leading to uncoupled subsystems. MDOF systems with such a restricted spectrum are rarely found in nature and engineered systems. Furthermore, the pairwise commutation condition $[N, K] = 0$, also demanded by Corollary 1, could place further restrictions on the last $n - 2m$ eigenvalues of Λ, making such a matrix K even more unlikely to arise in real-world MDOF systems. Likewise, the commutation conditions $[G, S] = [N, S] = 0$ restrict the spectrum of S in a similar manner. Thus, though only six pairwise commutation conditions in Corollary 1 need to be satisfied, from an applications standpoint they do not provide a useful set of conditions for uncoupling real-world MDOF systems, except perhaps in rare, very special structural and mechanical systems. Example 4 in the next section illustrates this remark.

3.1 Uncoupling Various Categories of Linear MDOF Structural and Mechanical Systems: A Unified Approach.

The n&s conditions given in (91) permit the simultaneous orthogonal quasi-diagonalization of the four matrices S, G, K, and N, two of which are symmetric and two of which are skew-symmetric. They allow us to obtain the n&s conditions, very simply, for all of the different categories of linear MDOF systems described in Sec. 1. One only needs to recall that the zero matrix commutes with all matrices. We refer to the matrix K as the "potential" matrix, a system that has $N = 0$ in (12) as a "potential system," and the matrix $R = K + N$ as the "stiffness" matrix.

Result 3. For each of the different categories of linear MDOF systems described by (12) the n&s conditions for an orthogonal matrix Q to exist, so that they can be maximally uncoupled through simultaneous quasi-diagonalization by the real coordinate transformation $x = Qp$ can be obtained using the set of n&s conditions given in (91). This is summarized in the table below for some systems commonly encountered in nature and in engineered systems.

MDOF system	System description in (12)	Necessary and sufficient conditions for maximal uncoupling	Equation of motion
Undamped potential system	$S=0,\ G=0,\ N=0$		$\ddot{p}+\Lambda p=Q^T f(t))$
Potential system with the symmetric damping matrix system	$G=0,\ N=0$	$[K, S]=0$	$\ddot{p}+\Sigma\dot{p}+\Lambda p=Q^T f(t)$
Gyroscopic purely circulatory system	$S=0,\ K=0$	$[G, N]=0$	$\ddot{p}+\Gamma\dot{p}+\mathrm{N}p=Q^T f(t)$
Gyroscopic potential system	$S=0,\ N=0$	$[K, G^2]=0,\ [K, KGK]=0$	$\ddot{p}+\Gamma\dot{p}+\Lambda p=Q^T f(t)$
Undamped system with arbitrary stiffness matrix	$S=0,\ G=0$	$[K, N^2]=0,\ [K, NKN]=0$	$\ddot{p}+(\Lambda+\mathrm{N})p=Q^T f(t)$
Gyroscopic nonconservative system with an arbitrary stiffness matrix	$S=0$	$[G, N]=0,\ [K, GN]=0,$ $[K, GKG]=0,\ [K, NKN]=0,$ $[K, G^2]=0,\ [K, N^2]=0$	$\ddot{p}+\Gamma\dot{p}+(\Lambda+\mathrm{N})p=Q^T f(t)$
Damped nonconservative system with an arbitrary stiffness matrix and an symmetric damping matrix	$G=0$	$[K, S]=0,\ [K, NKN]=0,$ $[K, NSN]=0,\ [S, NKN]=0,$ $[S, NSN]=0,\ [K, N^2]=0,\ [S, N^2]=0$	$\ddot{p}+\Sigma\dot{p}+(\Lambda+\mathrm{N})p=Q^T f(t)$
Nonconservative purely circulatory system with an arbitrary damping matrix and its dual damped gyroscopic nonconservative system with a purely circulatory stiffness matrix and a symmetric damping matrix	$K=0$	$[G, N]=0,\ [S, GN]=0,\ [S, GSG]=0,$ $[S, NSN]=0,\ [S, G^2]=0,\ [S, N^2]=0$	$\ddot{p}+(\Sigma+\Gamma)\dot{p}+\mathrm{N}p=Q^T f(t)$
Damped potential system with arbitrary damping matrix and its dual gyroscopic potential system with a symmetric damping matrix	$N=0$	$[K, S]=0,\ [K, GKG]=0,$ $[K, GSG]=0,\ [S, GKG]=0$ $[S, GSG]=0,\ [K, G^2]=0,\ [S, G^2]=0$	$\ddot{p}+(\Sigma+\Gamma)\dot{p}+\Lambda p=Q^T f(t)$
Damped nonconservative system with arbitrary damping and stiffness matrices		Commutation conditions in the set (91)	$\ddot{p}+(\Sigma+\Gamma)\dot{p}+(\Lambda+\mathrm{N})p=Q^T f(t)$

Proof. The n&s conditions are obtained by setting the appropriate matrices to zero in (91). For example, the n&s conditions for a gyroscopic potential system with a symmetric damping matrix are obtained by setting $N=0$ in (91), as shown in the second-last row of the table. The last column gives the maximally uncoupled equation of motion of the dynamical system in the principal coordinate p. When $G \neq 0$ (in our notation), the quasi-diagonal forms for Σ, Γ, Λ, and N are given in (93), and when $N \neq 0$ (in our notation) they are given in (98). The former is useful when $N=0$, the latter when $G=0$. The n&s commutation conditions for other linear MDOF systems not shown in this table can be similarly obtained using (91), and the quasi-diagonal forms for Σ, Γ, Λ, and N from (93) and (98). ■

From the last column of the table and the quasi-diagonal forms given in (93) and (98), we see that, in general, linear MDOF systems of the form given in (12) (maximally) uncouple into independent (real) subsystems each of at most two degrees-of-freedom when the appropriate n&s conditions are satisfied, except for the category of potential systems and classically damped MDOF systems (shown in the first two rows), which lead to independent single-degree-of-freedom subsystems. The n&s conditions, and the quasi-diagonal forms that lead to the uncoupling of some specific categories of linear MDOF systems into independent subsystems, have been investigated earlier in Refs. [3–7]. The present article obtains these results in a straightforward manner, providing a unified approach for (a) understanding under what conditions various categories of linear MDOF dynamical can be uncoupled through simultaneous orthogonal quasi-diagonalization and (b) providing the explicit nature of the resulting uncoupled subsystems. The coordinate change that brings about this uncoupling uses orthogonal matrices, conferring robustness to computational methods that take advantage of the uncoupled forms obtained when, of course, the n&s conditions are satisfied.

Remark 16. Since we have four matrices S, G, K, and N, from a mathematical standpoint, there will be 15 qualitatively different linear MDOF dynamical systems, excluding the trivial system in which all four matrices are zero. However, as noted in Sec. 1 and illustrated in the table, some of these systems may have dual counterparts. Although their mathematical descriptions are identical, their physical nature is significantly different, shaped by our understanding of the distinctive characteristics and origins of the forces involved. Though they have the same mathematical description, these dual systems would therefore qualify as belonging to different dynamical categories of vibratory systems from a physical viewpoint, as is the current practice [9].

We next summarize some known results [3–7] in the following lemma, which will be used in the next result.

LEMMA 9. *Consider the four (possible) matrix triplets* $\{S, G, K\}$, $\{S, N, K\}$, $\{K, G, N\}$, *and* $\{S, G, N\}$, *formed from the matrix quadruplet* $\{S, G, K, N\}$. *The following results are known from earlier studies [5–7].*

(a) *For a real orthogonal matrix* Q *to exist so that the matrix triplet* $\{S, G, K\}$, *which contains two symmetric matrices and one skew-symmetric matrix, can be simultaneously quasi-diagonalized, the set of n&s commutation conditions are*

$$[K, S]=[K, G^2]=[S, G^2]=0 \text{ and}$$
$$[K, GKG]=[K, GSG]=[S, GKG]=[S, GSG]=0$$

(b) *Similarly, for a real orthogonal matrix* Q *to exist so that the matrix triplet* $\{S, N, K\}$, *which also contains two symmetric matrices and one skew-symmetric matrix, can be simultaneously quasi-diagonalized, the set of n&s commutation conditions are (replace G by N in (a))*

$$[K, S]=[K, N^2]=[S, N^2]=0 \text{ and}$$
$$[K, NKN]=[K, NSN]=[S, NKN]=[S, NSN]=0$$

(c) *For a real orthogonal matrix* Q *to exist so that the matrix triplet* $\{K, G, N\}$, *which contains two skew-symmetric matrices and one symmetric matrix, can be simultaneously quasi-diagonalized, the set of n&s commutation conditions are*

$$[G, N]=[K, G^2]=[K, N^2]=0 \text{ and}$$
$$[K, GN]=[K, GKG]=[K, NKN]=0$$

(d) *Similarly, for a real orthogonal matrix Q to exist so that the matrix triplet $\{S, G, N\}$, which also contains two skew-symmetric matrices and one symmetric matrix, can be simultaneously quasi-diagonalized, the set of n&s commutation conditions are (replace K by S in (c))*

$$[G, N] = [S, G^2] = [S, N^2] = 0 \text{ and}$$
$$[S, GN] = [S, GSG] = [S, NSN] = 0$$

As mentioned before, these n&s conditions that were obtained earlier in Refs. [3–7] can be handily obtained from (91) by setting $N = 0$, $G = 0$, $S = 0$, and $K = 0$, as we did in the table, to get the results in parts (a), (b), (c), and (d), respectively. However, these references also contain several other results germain to real-world civil, aerospace, and mechanical engineering applications, which we will use in the following subsection. We note that the n&s conditions given in Ref. [5] have already been used to prove our central Theorems 1 and 3 (see Lemma 6).

Result 4. A real orthogonal matrix Q exists so that the four matrices S, G, K, and N can be simultaneously quasi-diagonalized by it, if and only if the four possible matrix triplets formed from these four matrices, namely, $\{S, G, K\}$, $\{S, N, K\}$, $\{K, G, N\}$, $\{S, G, N\}$, can each be (independently) simultaneously quasi-diagonalized by real orthogonal matrices.

Proof. Assume first that for a given matrix quadruplet $\{S, G, K, N\}$, the four possible triplets can be individually quasi-diagonalized by real orthogonal matrices. The n&s conditions for this to happen for each triplet are specified in Lemma 9. The orthogonal matrices that simultaneously quasi-diagonalize each of the four different matrix triplets are different, since the n&s commutation conditions for each triplet's orthogonal quasi-diagonalization are different, as seen in this lemma. However, it is easy to verify that the union of the set of n&s conditions given in Lemma 9(a)–(d) for each of these four triplets to be individually orthogonally quasi-diagonalized is the set shown in (91).

Hence, if each of four matrix triplets can be individually simultaneously quasi-diagonalized, all the n&s conditions for the matrix quadruplet $\{S, G, K, N\}$ to be orthogonally quasi-diagonalized are automatically satisfied. So the simultaneous orthogonal quasi-diagonalization of the four separate matrix triplets implies the simultaneous orthogonal quasi-diagonalization of the matrices S, G, K, and N.

On the other hand, if we assume S, G, K, and N can be simultaneously quasi-diagonalized by a real orthogonal matrix Q, this would imply that the four triplets must be quasi-diagonalizable by Q too. ■

Remark 17. Result 4 states that simultaneous orthogonal quasi-diagonalization of each of the four possible matrix triplets $\{S, G, K\}$, $\{S, N, K\}$, $\{K, G, N\}$, $\{S, G, N\}$ formed from the quadruplet $\{S, G, K, N\}$ is necessary and sufficient for the simultaneous orthogonal quasi-diagonalization of the matrix of S, G, K, and N. This appears somewhat intuitive.

Similarly, one might then conjecture that the orthogonal quasi-diagonalization of each of the three possible matrix doublets $\{S, G\}$, $\{S, K\}$, and $\{G, K\}$ formed from the matrix triplet $\{S, G, K\}$ would be n&s for the simultaneous quasi-diagonalization of the matrices S, G, and K. However, this is not true.

This is because the n&s conditions for $\{S, G\}$ to be quasi-diagonalized are $[S, G^2] = [S, GSG] = 0$; the n&s condition for $\{S, K\}$ to be diagonalized is $[S, K] = 0$; and the n&s conditions for $\{G, K\}$ to be quasi-diagonalized are $[K, G^2] = [K, GKG] = 0$. The union of these three sets of n&s conditions does not cover the n&s conditions for the simultaneous orthogonal quasi-diagonalization of the matrix triplet $\{S, G, K\}$ given in the second-last row of the table and Lemma 9(a)—the n&s commutation conditions $[K, GSG] = 0$ and $[S, GKG] = 0$ that are required for the simultaneous orthogonal quasi-diagonalization of the matrix triplet $\{S, G, K\}$ are not in the union of the three sets of n&s conditions for the three doublets. Likewise, the same conclusion can be drawn for the other triplets $\{S, N, K\}$, $\{K, G, N\}$, and $\{S, G, N\}$.

We next consider the case when $n \leq 2m + 2$.

Result 5. When the dimension n of the n by n matrices, S, $G \neq 0$, K, and N, is restricted so that $n \leq 2m + 2$ (recall, $0 < Rank\,(G) = 2m \leq n$), then the necessary and sufficient conditions for a real orthogonal matrix Q to exist such that the MDOF dynamical system described by (12) can be uncoupled through simultaneous quasi-diagonalization, using the orthogonal coordinate transformation $x = Qp$, into independent subsystems that have at most two degrees-of-freedom are

$$\begin{aligned}
&[G, N] = 0, && [K, S] = 0, \\
&[K, GKG] = 0, && [K, GSG] = 0, \\
&[S, GKG] = 0, && [S, GSG] = 0, \\
&[K, GN] = 0, && [S, GN] = 0, \\
&[K, G^2] = 0, && [S, G^2] = 0
\end{aligned} \tag{102}$$

The equation describing the uncoupled system in terms of the principal coordinate p is

$$\ddot{p} + (\Sigma + \Gamma)\dot{p} + (\Lambda + \mathrm{N})p = Q^T f(t) \tag{103}$$

where

$$\begin{aligned}
\Sigma &= Q^T SQ = \mathrm{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_n) \\
\Gamma &= Q^T GQ = \mathrm{diag}\,(\beta_1 J_2, \ldots, \beta_m J_2) \text{ for } n = 2m \\
&= \mathrm{diag}\,(\beta_1 J_2, \ldots, \beta_m J_2, 0) \text{ for } n = 2m + 1 \\
&= \mathrm{diag}\,(\beta_1 J_2, \ldots, \beta_m J_2, 0, 0) \text{ for } n = 2m + 2 \\
\Lambda &= Q^T KQ = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)
\end{aligned} \tag{104}$$

and

$$\begin{aligned}
\mathrm{N} &= Q^T NQ = \mathrm{diag}\,(\nu_1 J_2, \ldots, \nu_m J_2) \text{ for } n = 2m \\
&= \mathrm{diag}\,(\nu_1 J_2, \ldots, \nu_m J_2, 0) \text{ for } n = 2m + 1 \\
&= \mathrm{diag}\,(\nu_1 J_2, \ldots, \nu_m J_2, v_{m+1} J_2) \text{ for } n = 2m + 2
\end{aligned}$$

where $\beta_j > 0$, $j = 1, \ldots, m$, and all the v_j's, λ_j's, and σ_j's are real numbers.

Each uncoupled, independent, real two-degree-of-freedom subsystem in (103) has the specific matrix structure given in (94).

Proof. Using Theorem 2 and following along the same lines as the proof of Result 1, we arrive at (103) and (104) along with the n&s conditions in (102). ■

We illustrate this by the following example.

Example 3. We consider same MDOF system described in Example 1, except that now we use the 6 by 6 matrix

$$G = \begin{bmatrix}
0 & -1.1082 & 0.2787 & 0.4000 & -0.4926 & -1.9147 \\
1.1082 & 0 & -1.1973 & -1.1241 & -0.6156 & -0.0085 \\
-0.2787 & 1.1973 & 0 & -0.6894 & -0.7312 & -0.0900 \\
-0.4000 & 1.1241 & 0.6894 & 0 & -0.4880 & -0.5059 \\
0.4926 & 0.6156 & 0.7312 & 0.4880 & 0 & -0.5363 \\
1.9147 & 0.0085 & 0.0900 & 0.5059 & 0.5363 & 0
\end{bmatrix}$$

The matrices S, K, and N, as well as the force four-vector $f(t)$, are the same as in Example 1. The spectrum of G is $\{\pm 2i, \pm 2.5i, 0, 0\}$ and $\mathrm{Rank}(G) = 4$, so we have $m = 2$ with $n = 6$. Observe that now $n = 2m + 2$, and Result 5 is therefore applicable. For simultaneous orthogonal quasi-diagonalization of the four matrices S, G, K, and N, we therefore require them to satisfy the commutation conditions in (102), which are a much smaller set of conditions than required in Example 1. A short computation shows that these 10 commutation conditions are satisfied, and therefore Result 5 is applicable.

Upon using the real coordinate transformation $x = Qp$, with

$$Q = \begin{bmatrix} 0.2004 & 0.5170 & -0.6287 & 0.5175 & 0.1704 & -0.0200 \\ 0.3679 & 0.5930 & 0.6025 & 0.1183 & -0.3568 & 0.0933 \\ 0.4750 & 0.1632 & 0.0514 & -0.4905 & 0.5768 & -0.4146 \\ 0.4896 & -0.1213 & -0.3002 & -0.3604 & -0.1371 & 0.7119 \\ 0.4640 & -0.3541 & -0.2078 & 0.0888 & -0.5703 & -0.5320 \\ 0.3791 & -0.4630 & 0.3252 & 0.5830 & 0.4086 & 0.1713 \end{bmatrix}$$

we obtain the following three independent, uncoupled subsystems, each having two degrees-of-freedom:

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 0.2164 & 2 \\ -2 & 0.2853 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} 0.1641 & 0 \\ 0 & 0.8530 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.2004 & 0.3679 \\ 0.5170 & 0.5930 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

$$\begin{bmatrix} \ddot{p}_3 \\ \ddot{p}_4 \end{bmatrix} + \begin{bmatrix} 0.4958 & 2.5 \\ -2.5 & 0.3771 \end{bmatrix} \begin{bmatrix} \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} + \begin{bmatrix} 2.9583 & 0 \\ 0 & 1.7714 \end{bmatrix} \begin{bmatrix} p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} -0.6287 & 0.6025 \\ 0.5175 & 0.1183 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

and

$$\begin{bmatrix} \ddot{p}_5 \\ \ddot{p}_6 \end{bmatrix} + \begin{bmatrix} 0.6094 & 0 \\ 0 & 0.8659 \end{bmatrix} \begin{bmatrix} \dot{p}_5 \\ \dot{p}_6 \end{bmatrix} + \begin{bmatrix} 4.0939 & 1 \\ -1 & 6.6593 \end{bmatrix} \begin{bmatrix} p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} 0.1704 & -0.3568 \\ -0.0200 & 0.0933 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

We next go back to Corollary 1 which considers the pairwise commutation of the four matrices S, G, K, and N for their simultaneous orthogonal quasi-diagonalization. Remark 15 says that these pairwise commutation conditions, though only 6 in number, restrict these four matrices much more than the commutation conditions given in (91). That is, there are matrix quadruplets $\{S, G, K, N\}$ that do not pairwise commute and therefore do not satisfy the requirements of Corollary 1, yet they can be simultaneously orthogonally quasi-diagonalized, because they satisfy (91). Loosely speaking, the "number" of quadruplets, $\{S, G, K, N\}$, that satisfy (91) is far greater than the "number" of quadruplets that satisfy pairwise commutation of these four matrices. We illustrate this in the next example using matrices that satisfy the commutation conditions given in Result 5.

Example 4. Consider the simple example of a four-degree-of-freedom system described by (12) in which

$$S = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2.28 & -0.96 \\ 0 & 0 & -0.96 & 1.72 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 & -2.64 & 0.48 \\ 0 & 0 & 0.48 & -2.36 \\ 2.64 & -0.48 & 0 & 0 \\ -0.48 & 2.36 & 0 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 2.64 & -0.48 & 0 & 0 \\ -0.48 & 2.36 & 0 & 0 \\ 0 & 0 & 2.28 & -0.96 \\ 0 & 0 & -0.96 & 1.72 \end{bmatrix}, N = \begin{bmatrix} 0 & 0 & 0.92 & -1.44 \\ 0 & 0 & -1.44 & 0.08 \\ -0.92 & 1.44 & 0 & 0 \\ 1.44 & -0.08 & 0 & 0 \end{bmatrix}$$

and

$$f(t) = \begin{bmatrix} f_1(t) & f_2(t) & f_3(t) & f_4(t) \end{bmatrix}^T$$

Again, since $Rank(G) = 4$ (i.e., $m = 2$) and $n = 4$, we have $n = 2m + 2$; therefore, Result 5 is applicable. The n&s conditions for the system described by (12) so it can be transformed to a quasi-diagonal form are then given in (102). An easy computation shows that these four matrices satisfy all these conditions. Hence, the system can be uncoupled into two independent two-degree-of-freedom subsystems. The orthogonal matrix

$$Q = \frac{1}{5} \begin{bmatrix} 0 & 3 & 4 & 0 \\ 0 & 4 & -3 & 0 \\ 3 & 0 & 0 & -4 \\ 4 & 0 & 0 & 3 \end{bmatrix}$$

generates the real orthogonal coordinate transformation $x = Qp$, which uncouples the system to yield the two independent, two-degree-of-freedom subsystems, each in the form (94), as

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.6f_3(t) + 0.8f_4(t) \\ 0.6f_1(t) + 0.8f_2(t) \end{bmatrix}$$

and

$$\begin{bmatrix} \ddot{p}_3 \\ \ddot{p}_4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 0.8f_1(t) - 0.6f_2(t) \\ -0.8f_3(t) + 0.6f_4(t) \end{bmatrix}$$

Observe that the pairwise commutation conditions stated in Corollary 1 demand, among others, that $[K, G] = 0$, $[K, N] = 0$,

$[S, G] = 0$, and $[S, N] = 0$. None of these four conditions are met by the matrices K, G, S, and N in this example. This shows that the ten n&s conditions in (102) are less restrictive on the four matrices S, G, K, and N than the six placed on them by the requirement that they commute pairwise.

Result 6. When the dimension n of the n by n matrices, S, G, K, and $N \neq 0$, is restricted so that $n \leq 2m+2$ (recall, $0 < Rank(N) = 2m \leq n$), then the necessary and sufficient conditions for a real orthogonal matrix Q to exist such that the MDOF dynamical system described by (12) can be uncoupled, by simultaneous quasi-diagonalization using the orthogonal coordinate transformation $x = Qp$, into independent subsystems that have at most two degrees-of-freedom are

$$\begin{aligned} &[G, N] = 0, &&[K, S] = 0, \\ &[K, NKN] = 0, &&[K, NSN] = 0, \\ &[S, NKN] = 0, &&[S, SNS] = 0, \\ &[K, GN] = 0, &&[S, GN] = 0, \\ &[K, N^2] = 0, &&[S, N^2] = 0 \end{aligned} \tag{105}$$

The equation describing the uncoupled system in terms of the principal coordinate p is

$$\ddot{p} + (\Sigma + \Gamma)\dot{p} + (\Lambda + \mathrm{N})p = Q^T f(t) \tag{106}$$

where

$$\begin{aligned} \Sigma &= Q^T S Q = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \\ \Gamma &= Q^T G Q = \mathrm{diag}(\beta_1 J_2, \ldots, \beta_m J_2) \text{ for } n = 2m \\ &= \mathrm{diag}(\beta_1 J_2, \ldots, \beta_m J_2, 0) \text{ for } n = 2m+1 \\ &= \mathrm{diag}(\beta_1 J_2, \ldots, \beta_m J_2, \beta_{m+1} J_2) \text{ for } n = 2m+2 \\ \Lambda &= Q^T K Q = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \end{aligned} \tag{107}$$

and

$$\begin{aligned} \mathrm{N} = Q^T N Q &= \mathrm{diag}(\nu_1 J_2, \ldots, \nu_m J_2) \text{ for } n = 2m \\ &= \mathrm{diag}(\nu_1 J_2, \ldots, \nu_m J_2, 0) \text{ for } n = 2m+1 \\ &= \mathrm{diag}(\nu_1 J_2, \ldots, \nu_m J_2, 0, 0) \text{ for } n = 2m+2 \end{aligned}$$

where $\nu_j > 0$, $j = 1, \ldots, m$, and all the β_j's, λ_j's, and σ_j's are real numbers.

Each uncoupled, independent, real two-degree-of-freedom subsystem in (106) has the specific matrix structure given in (94).

Proof. Using Theorem 4 and following along the same lines as the proof of Result 1, we arrive at (106) and (107) along with the n&s conditions in (105). ∎

Remark 18. As seen in Results 1 and 2, which are more general, there are 16 n&s conditions in (91) for a real orthogonal matrix Q to exist so that the matrix quadruplet $\{S, G, K, N\}$ can be simultaneously quasi-diagonalized. On the other hand, Results 5 and 6 are restricted to the case when $n \leq 2m+2$ and they each require only the ten n&s conditions given in (102) and (105), respectively, for simultaneous orthogonal quasi-diagonalization of the quadruplet. From this, we deduce that the remaining six n&s conditions not mentioned in (102) and (105) are automatically satisfied when $n \leq 2m+2$. That this is indeed so can be seen from the quasi-diagonal forms given in (104) and (107), respectively, that guarantee this. For example, using (104) in Result 5, we find that $[K, NKN] = Q[Q^T KQ, Q^T(NKN)Q]Q^T = Q[\Lambda, \mathrm{N}\Lambda\mathrm{N}]Q^T = 0$. The last equality follows from Remark 4, part 3, since Λ and $\mathrm{N}\Lambda\mathrm{N}$ are diagonal matrices. Similarly, when (102) is true, the commutation condition $[K, N^2] = 0$ is automatically satisfied.

The reduction in the number of commutation conditions from 16, in general, to 10 for simultaneous orthogonal quasi-diagonalization when either the matrix $G \neq 0$ and $n \leq 2m+2$, or when $N \neq 0$ and $n \leq 2m+2$, includes the commonly found case where either G and/or N has full rank. However, having to satisfy 10 conditions is still quite a large number that the four matrices S, G, K, and N need to satisfy to uncouple (12) maximally using simultaneous quasi-diagonalization. We now explore how we might reduce this number by considering more information, which we often possess in systems that commonly arise in nature and in engineering, about the matrices S, G, K, and N beyond what we have assumed till now.

3.2 Reduction in the Number of N&S Commutation Conditions. We begin by assuming that the spectra of the skew-symmetric matrices G and N are such that their nonzero eigenvalues are distinct in each of them. This is a circumstance that commonly arises in naturally occurring MDOF systems as well as those that are engineered because having distinct nonzero eigenvalues is generic for these matrices.

Result 7A. If the nonzero eigenvalues of G are distinct, and the nonzero eigenvalues of N are distinct, then an orthogonal matrix Q exists such that the quadruplet $\{S, G, K, N\}$ can be simultaneously quasi-diagonalized into the forms given in (14)–(17) if and only if the following six commutation conditions are satisfied:

$$\begin{aligned} &[K, S] = 0, [G, N] = 0, [K, G^2] = 0, [K, N^2] = 0, \\ &[S, G^2] = 0, [S, N^2] = 0 \end{aligned} \tag{108}$$

These are the n&s conditions for (12) to be uncoupled by the orthogonal coordinate change $x = Qp$ to yield the uncoupled form

$$\ddot{p} + (\Sigma + \Gamma)\dot{p} + (\Lambda + \mathrm{N})p = Q^T f(t) \tag{109}$$

which is composed of at most two degrees-of-freedom independent subsystems.

Proof. We consider the four matrix triplets $\{S, G, K\}$, $\{S, N, K\}$, $\{K, G, N\}$, and $\{S, G, N\}$ when G and N each have distinct nonzero eigenvalues. The n&s conditions for each triplet to be orthogonally quasi-diagonalized are as follows.

(a) For the triplet $\{S, G, K\}$, as shown in Ref. [5], the n&s conditions are:

$$[S, G^2] = 0, [K, G^2] = 0, \text{ and } [K, S] = 0 \tag{110}$$

(b) For the triplet $\{S, N, K\}$, replacing G by N in (110), the n&s conditions are:

$$[S, N^2] = 0, [K, N^2] = 0, \text{ and } [K, S] = 0 \tag{111}$$

(c) For the triplet $\{K, G, N\}$, as shown in Ref. [6], the n&s conditions are:

$$[G, N] = 0, [K, G^2] = 0, [K, N^2] = 0 \tag{112}$$

(d) For the triplet $\{S, G, N\}$, replacing K by S in (112), the n&s conditions are:

$$[G, N] = 0, [S, G^2] = 0, [S, N^2] = 0 \tag{113}$$

Thus, according to Result 4, for the quadruplet $\{S, G, K, N\}$ to be reduced by a real orthogonal transformation Q to a quasi-diagonal form, the set of n&s conditions are the union of sets in (110)–(113), as shown in (108).

Using the coordinate transformation $x = Qp$ as before, the form (109) is obtained from (12), leading to at most two degrees-of-freedom independent subsystems in the principal coordinate p. ∎

Result 7B. If the nonzero eigenvalues of G are distinct, and the nonzero eigenvalues of N are distinct, then an orthogonal matrix Q exists such that the quadruplet $\{S, G, K, N\}$ can be simultaneously quasi-diagonalized into the forms given in (19)–(22) or (26)–(29) if and only if the following four commutation conditions are satisfied:

$$\begin{aligned}&\text{(a)} \quad \text{when} \quad G \neq 0 \text{ and } n \leq 2m+2,\\ &[K, S] = 0, [G, N] = 0, [K, G^2] = 0, [S, G^2] = 0\end{aligned} \tag{114}$$

$$\begin{aligned}&\text{(b)} \quad \text{when} \quad N \neq 0 \text{ and } n \leq 2m+2,\\ &[K, S] = 0, [G, N] = 0, [K, N^2] = 0, [S, N^2] = 0\end{aligned} \tag{115}$$

These are the n&s conditions for (12) to be uncoupled through simultaneous quasi-diagonalization by the orthogonal coordinate change $x = Qp$ to yield the uncoupled form

$$\ddot{p} + (\Sigma + \Gamma)\dot{p} + (\Lambda + \mathrm{N})p = Q^T f(t) \tag{116}$$

which is composed of at most two degrees-of-freedom independent subsystems.

Proof. When $G \neq 0$ (recall, $Rank(G) = 2m$), $[S, N^2] = 0$ and $[K, N^2] = 0$, and when $N \neq 0$, $[K, G^2] = 0$ and $[S, G^2] = 0$ (see Remark 18). Using Result 7A, the result follows. The quasi-diagonal forms for Σ, Γ, Λ, and N when $G \neq 0$ can be obtained from (104), and when $N \neq 0$ from (107). ∎

Often, the eigenvalues of the symmetric matrix $K(S)$ are distinct in many real-world systems in aerospace, civil, and mechanical engineering, this, again, being generic for symmetric matrices. This leads us to the following result.

Result 8A

(1) If all the eigenvalues of K are distinct, then a real orthogonal matrix Q exists such that S, $G \neq 0$, K, and N can be simultaneously orthogonally quasi-diagonalized into the forms given in (14)–(17) if and only if the following six commutation conditions are satisfied:

$$\begin{aligned}&[K, S] = 0, [K, GN] = 0, [K, G^2] = 0, [K, GKG] = 0,\\ &\qquad [K, N^2] = 0, [K, NKN] = 0\end{aligned} \tag{117}$$

(2) If all the eigenvalues of N are distinct, then a real orthogonal matrix Q exists such that S, G, K, and $N \neq 0$ can be simultaneously orthogonally quasi-diagonalized into the forms given in (26)–(29) if and only if the six commutation conditions in (117) are satisfied.

These are the n&s conditions for uncoupling (12), using the real coordinate transformation $x = Qp$ to obtain the uncoupled forms made up of independent subsystems each of which has at most two degrees-of-freedom.

Proof. (1) Since K is symmetric, it can be diagonalized by an orthogonal matrix Q so that the matrix Q^TKQ is a diagonal matrix that has the distinct eigenvalues of K along its diagonal. The columns of the matrix Q are the n orthonormal eigenvectors of K, and so Q is unique, except for interchanges among its columns. We use this idea to show that the n&s conditions for the quadruplet $\{S, G, K, N\}$ with $G \neq 0$ to be simultaneously orthogonally quasi-diagonalized are those shown in (117). We prove the sufficiency first.

Consider the first commutation condition in (117). Since $[K, S] = Q[Q^TKQ, Q^TSQ]Q^T = 0$ and Q^TKQ is a diagonal matrix with distinct elements along its diagonal, the matrix Q^TSQ must be diagonal (Remark 4, part 5). Hence, Q simultaneously diagonalizes K and S. At present, we leave the ordering of the eigenvectors in Q unspecified, though we note that the simultaneous diagonalization K and S results from *any* arbitrary ordering of the n eigenvectors of K contained in the columns of the matrix Q.

Now consider the remaining five commutation conditions in (117), which are the n&s conditions for the triplet $\{K, G \neq 0, N\}$ to be simultaneously orthogonally quasi-diagonalized when K has distinct eigenvalues (see Ref. [6], Result 2(a)). We know that the only matrix that can diagonalize the member K of this triplet is the matrix Q whose columns are the eigenvectors (now listed in some particular order) of K. Hence, the matrix Q_1 that simultaneously orthogonally quasi-diagonalized $\{K, G \neq 0, N\}$ *must* comprise of the n eigenvectors of Q in some specific order. And this matrix Q_1 will also simultaneously diagonalize S since this happens with any arbitrary ordering of the eigenvectors of Q, as noted before. Hence, Q_1 simultaneously orthogonally quasi-diagonalizes the quadruplet $\{S, G, K, N\}$.

The necessity of the conditions in (117) is trivial to show since the existence of a real orthogonal matrix Q_1 that simultaneously quasi-diagonalized $\{S, G, K, N\}$ leads to quasi-diagonal forms shown in (93) from which satisfaction of the commutation relation follows.

The use of the coordinate transformation $x = Qp$ uncouples (12) to yield the uncoupled form (109) in the principal coordinate p.

(2) We get the same result by replacing G by N in the argument above. The set of commutation conditions in (117) remain unaltered. ∎

Remark 19. As in the proof of Result 7A, we could apply Result 4 and consider the four individual matrix triplets $\{S, G, K\}$, $\{S, N, K\}$, $\{K, G, N\}$, and $\{S, G, N\}$, assuming that K has distinct eigenvalues. The n&s conditions for each triplet to be orthogonally simultaneously quasi-diagonalized when K has distinct eigenvalues are as follows.

(a) For the triplet $\{S, G, K\}$, as shown in Ref. [5], the n&s conditions are

$$[K, S] = 0, [K, GKG] = 0, [K, G^2] = 0$$

(b) For the triplet $\{S, N, K\}$, replacing G by N above, the n&s conditions are

$$[K, S] = 0, [K, NKN] = 0, [K, N^2] = 0$$

(c) For the triplet $\{K, G, N\}$, as shown in Ref. [6], the n&s conditions become

$$\begin{aligned}&[K, GN] = 0, [K, G^2] = 0, [K, GKG] = 0,\\ &[K, N^2] = 0, [K, NKN] = 0\end{aligned}$$

(d) According to Lemma 9(d), for the triplet $\{S, G, N\}$, the n&s conditions are

$$\begin{aligned}&[G, N] = 0, [S, GN] = 0, [S, G^2] = 0,\\ &[S, GSG] = 0, [S, N^2] = 0, [S, NSN] = 0\end{aligned}$$

Taking the union of the four sets of n&s conditions listed in (a)–(d), we get the following set:

$$\begin{aligned}&[K, S] = 0, [K, N^2] = 0, [K, G^2] = 0, [K, GN] = 0,\\ &[S, G^2] = 0, [S, N^2] = 0, [G, N] = 0, [K, GKG] = 0, [K, NKN] = 0,\\ &[S, GSG] = 0, [S, NSN] = 0, [S, GN] = 0\end{aligned}$$

of n&s commutation conditions for an orthogonal matrix, say Q_1, to exist such $\{S, G, K, N\}$ can be simultaneously quasi-diagonalized. However, these 12 commutation conditions are not all independent when one considers that the eigenvalues of K are distinct, leaving just the six independent conditions in Result 8A.

Corollary 2. *If all the eigenvalues of K are distinct, then a real orthogonal matrix Q exists such that the quadruplet $\{S, G, K, N\}$*

can be simultaneously orthogonally quasi-diagonalized into the forms given in (14)–(17) or (26)–(29) if and only if:

(a) *when the nonzero eigenvalues of G are distinct, the following five commutation conditions are satisfied:*

$$[K, S] = 0, [K, GN] = 0, [K, G^2] = 0, [K, N^2] = 0, [K, NKN] = 0 \quad (118)$$

(b) *when the nonzero eigenvalues of N are distinct, the following five commutation conditions are satisfied:*

$$[K, S] = 0, [K, GN] = 0, [K, G^2] = 0, [K, N^2] = 0, [K, GKG] = 0 \quad (119)$$

(c) *when the nonzero eigenvalues of G are distinct and the nonzero eigenvalues of N are distinct, the following four commutation conditions are satisfied:*

$$[K, S] = 0, [K, GN] = 0, [K, G^2] = 0, [K, N^2] = 0 \quad (120)$$

These are the n&s conditions for uncoupling (12) to obtain the uncoupled form (109) that is made up of independent subsystems each of which has at most two degrees-of-freedom.

Proof. This is because when the nonzero eigenvalues of G are distinct, $[K, G^2] = 0$ implies $[K, GKG] = 0$, and when the nonzero eigenvalues of N are distinct, $[K, N^2] = 0$ implies $[K, NKN] = 0$ [3].

Remark 20. If the eigenvalues of K are distinct, the nonzero eigenvalues of G are distinct and $[K, G^2] = 0$, then the condition $[K, GN] = 0$ is equivalent to the condition $[G, N] = 0$. This follows easily from Ref. [6] (Lemmas 5 and 6). Therefore, in (118)–(120), the condition $[K, GN] = 0$ can be replaced by $[G, N] = 0$.

Result 8B. If all the eigenvalues of K are distinct, then an orthogonal matrix Q exists such that the quadruplet $\{S, G, K, N\}$ can be simultaneously orthogonally quasi-diagonalized into the forms given in (19)–(22) or (26)–(29) if and only if the following four commutation conditions are satisfied:

(a) when $G \neq 0$ and $n \leq 2m + 2$,

$$[K, S] = 0, [K, GN] = 0, [K, G^2] = 0, [K, GKG] = 0 \quad (121)$$

(b) when $N \neq 0$ and $n \leq 2m + 2$,

$$[K, S] = 0, [K, GN] = 0, [K, N^2] = 0, [K, NKN] = 0 \quad (122)$$

These are the n&s conditions for uncoupling (12) to obtain the uncoupled form (109) that is made up of independent subsystems each of which has at most two degrees-of-freedom.

Proof. We use (117) from Result 8A. Remark 18 shows that when $G \neq 0$, the last two commutation conditions in (117) are automatically satisfied. Similarly, when $N \neq 0$, the third and fourth commutation conditions are automatically satisfied. ■

Corollary 3. *If all the eigenvalues of K are distinct, then an orthogonal matrix Q exists such that the quadruplet* $\{S, G, K, N\}$ *can be simultaneously orthogonally quasi-diagonalized into the forms given in (19)–(22) or (26)–(29) if and only if the following three commutation conditions are satisfied:*

(a) when the nonzero eigenvalues of G are distinct and $n \leq 2m + 2$,

$$[K, S] = 0, [K, GN] = 0, [K, G^2] = 0 \quad \text{and} \quad (123)$$

(b) when the nonzero eigenvalues of N are distinct and $n \leq 2m + 2$,

$$[K, S] = 0, [K, GN] = 0, [K, N^2] = 0 \quad (124)$$

These are the n&s conditions for uncoupling (12) through simultaneous orthogonal quasi-diagonalization to obtain the uncoupled form (109) that comprises independent subsystems, each with at most two degrees-of-freedom.

Proof. As in Corollary 2, the reason for the reduction in the number of commutator conditions from (121) and (122) to (123) and (124), respectively, is that when the nonzero eigenvalues of G are distinct, $[K, G^2] = 0$ implies $[K, GKG] = 0$; similarly, when the nonzero eigenvalues of N are distinct, $[K, N^2] = 0$ implies $[K, NKN] = 0$.
The matrices Σ, Γ, Λ, and N for case (a) are given in (104); for case (b), in (107). ■

In the sets of conditions (123) and (124), $[K, GN] = 0$ can be replaced by $[G, N] = 0$ (see Remark 20).
The roles of K and S can be interchanged in Results 8A and 8B. This leads to the following two results.

Result 8C. If all the eigenvalues of S are distinct, then an orthogonal matrix Q exists such that the quadruplet $\{S, G, K, N\}$ can be simultaneously orthogonally quasi-diagonalized into the forms given in (14)–(17) or (26)–(29) if and only if the following six commutation conditions are satisfied:

$$[K, S] = 0, [S, GN] = 0, [S, G^2] = 0, [S, GSG] = 0, [S, N^2] = 0, [S, NSN] = 0 \quad (125)$$

These are the n&s conditions for uncoupling (12) to obtain the uncoupled form (109) that is made up of independent subsystems each of which has at most two degrees-of-freedom.

Result 8D. If all the eigenvalues of S are distinct, then an orthogonal matrix Q exists such that the quadruplet $\{S, G, K, N\}$ can be simultaneously orthogonally quasi-diagonalized into the forms given in (19)–(22) or (26)–(29) if and only if the following four commutation conditions are satisfied:

(a) when $G \neq 0$ (recall, $Rank\,(G) = 2m$) and $n \leq 2m + 2$,

$$[K, S] = 0, [S, GN] = 0, [S, G^2] = 0, [S, GKG] = 0 \quad (126)$$

(b) when $N \neq 0$ and $n \leq 2m + 2$,

$$[K, S] = 0, [S, GN] = 0, [S, N^2] = 0, [S, NSN] = 0 \quad (127)$$

These are the n&s conditions for uncoupling (12) to obtain the uncoupled form (109) that is made up of independent subsystems each of which has at most two degrees-of-freedom.

Remark 21. Likewise, in Corollaries 2 and 3, the roles of K and S can be interchanged.

Lemma 10

(a) If and only if $[K, G^2] = 0$ *and* $[K, GKG] = 0$, *there exists an orthogonal matrix Q such that the matrices K and* $G \neq 0$ *can be simultaneously quasi-diagonalized, with*

$$\Lambda = Q^T KQ = \Lambda = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \quad (128)$$

$$\Gamma = Q^T GQ = \Gamma = \text{diag}\,(\beta_1 J_2, \beta_2 J_2, \ldots, \beta_m J_2, 0_{n-2m}) \quad (129)$$

where the λ_i's *are real numbers, and* $\pm\beta_j i$, $j = 1, 2, \ldots, m$, *are the nonzero imaginary eigenvalues of G, and further*

(b) *when the nonzero eigenvalues of G are distinct, then* $[K, G^2] = 0$ *implies* $[K, GKG] = 0$. *Hence,* $[K, G^2] = 0$ *is the sole n&s condition for the simultaneous orthogonal quasi-diagonalization of K and G by the orthogonal matrix Q.*

(c) *When* $[K, G^2] = 0$ *and* $[K, GKG] = 0$, *then*

$$[K^j, G^{2l+1}K^uG^{2r+1}] = [K^j, G^{2l}(GK^uG)G^{2r}] = 0$$

for any non-negative integers j, l, u, and r.

Proof. Parts (a) and (b) are proved in Result 3(a) in Ref. [3].

We prove part (c) now. Noting that $J_2^2 = -I_2$, and using (128) and (129), for any non-negative integers u and l, we have the following three diagonal matrices:

$$\begin{aligned}
\Lambda^u &= Q^T K^u Q = \text{diag}\,(\lambda_1^u, \lambda_2^u, \ldots, \lambda_n^u)\\
\Gamma^{2l} &= Q^T G^{2l} Q = (-1)^l \text{diag}\,(\beta_1^{2l} I_2, \beta_2^{2l} I_2, \ldots, \beta_m^{2l} I_2, 0_{n-2m})\\
\Gamma^{2l+1} &= Q^T G^{2l+1} Q = (-1)^l \text{diag}\,(\beta_1^{2l+1} J_2, \beta_2^{2l+1} J_2, \ldots, \beta_m^{2l+1} J_2, 0_{n-2m})
\end{aligned}$$

The product matrix $\Xi := \Gamma^{2l+1}\Lambda^u\Gamma^{2r+1}$ is then a diagonal matrix, where r is a non-negative integer.

Noting that $Q^TQ = I$, we, therefore, find that for arbitrary non-negative integers j, l, u, and r

$$\begin{aligned}
[K^j, G^{2l+1}K^uG^{2r+1}] &= Q[Q^TK^jQ, Q^TG^{2l+1}K^uG^{2r+1}Q]Q^T\\
&= Q[\Lambda^j, \Gamma^{2l+1}\Lambda^u\Gamma^{2r+1}]Q^T\\
&= Q[\Lambda^j, \Xi]Q^T = 0
\end{aligned}$$

The last equality follows from Remark 4, part 3. ∎

Result 9. Let $0 < Rank\,(G) = 2m \le n$ $(G \neq 0)$ and

$$S = \sum_{j=0}^{n-1} a_j K^j \text{ and } N = \sum_{j=0}^{e} b_j G^{2j+1} \tag{130}$$

where the coefficients a_j's and b_j's are real constants, and e is a non-negative integer. Then the two conditions

$$[K, G^2] = 0 \text{ and } [K, GKG] = 0 \tag{131}$$

are n&s for an orthogonal matrix Q to exist such that (12) can be quasi-diagonalized and transformed by a real orthogonal coordinate transformation $x = Qp$ to the uncoupled form (109), with the matrices Σ, Γ, Λ, and N given in (93), yielding real, independent subsystems each of which has at most two degrees-of-freedom.

Proof. The other 14 necessary and sufficient commutation conditions in (91) are satisfied automatically when $[K, G^2] = 0$ and $[K, GKG] = 0$. A detailed proof is given in the Appendix. Therefore, by Result 1, a real orthogonal matrix Q exists such that the four matrices S, G, K, and N can be simultaneously quasi-diagonalized. The coordinate transformation $x = Qp$ in (12) then gives the uncoupled form (109), which yields (real) independent subsystems each of which has at most two degrees-of-freedom. ∎

Corollary 4. *Let* $S = \sum_{j=0}^{n-1} a_j K^j$ *and* $N = \sum_{j=0}^{e} b_j G^{2j+1}$ *where the coefficients* a_j'*s and* b_j'*s are real constants and e is a non-negative integer. When the nonzero eigenvalues of G are distinct, then there exists a real orthogonal matrix Q such that (12) can be quasi-diagonalized and transformed by a real orthogonal coordinate transformation* $x = Qp$ *to the uncoupled form (109) if and only if*

$$[K, G^2] = 0$$

The uncoupled form leads to independent subsystems each having at most two degrees-of-freedom.

Proof. By Lemma 10(b), when the nonzero eigenvalues of G are distinct, the commutation condition $[K, G^2] = 0$ implies the condition $[K, GKG] = 0$. Result 9 proves the corollary. Note that G cannot now be a zero matrix. ∎

Lemma 11. *Let* $[K, S] = 0$ *and the eigenvalues of K be distinct, then S can be expressed as* $S = \sum_{j=0}^{n-1} a_j K^j$ *[8]. Thus, the orthogonal matrix Q, which contains all the eigenvectors of K, simultaneously diagonalizes K and S; as stated before, the order of the eigenvectors in the columns of Q does not matter.*

Corollary 5. *Let K have distinct eigenvalues, and* $N = \sum_{j=0}^{e} b_j G^{2j+1}$ *where the coefficients* b_j'*s are real constants and e is a non-negative integer. When the nonzero eigenvalues of G are distinct, then there exists a real orthogonal matrix Q such that (12) can be quasi-diagonalized and transformed by a real orthogonal coordinate transformation* $x = Qp$ *to the uncoupled form (109) if and only if*

$$[K, S] = 0,\ [K, G^2] = 0$$

The uncoupled form leads to independent subsystems each having at most two degrees-of-freedom.

Proof. Using Corollary 4 and Lemma 11, the result follows. ∎

We next consider the case when G has distinct eigenvalues. Let us assume that an orthogonal matrix Q exists that simultaneously quasi-diagonalizes the matrices S, $G \neq 0$, K, and N. Hence $Q^TSQ = \Sigma$, $Q^TGQ = \Gamma$, $Q^TKQ = \Lambda$, and $Q^TNQ = \text{N}$.

Furthermore, since $0 < Rank\,(G) = 2m \le n$, i.e., $G \neq 0$, and since the eigenvalues of G are distinct

$$\begin{aligned}
Q^TGQ = \Gamma &= \text{diag}\,(\beta_1 J_2, \beta_2 J_2, \ldots, \beta_{n/2} J_2) \text{ for } n \text{ even}\\
&= \text{diag}\,(\beta_1 J_2, \beta_2 J_2, \ldots, \beta_{(n-1)/2} J_2, 0) \text{ for } n \text{ odd}
\end{aligned} \tag{132}$$

in which the β_j's > 0 are now all distinct. We also have

$$\begin{aligned}
Q^TNQ = \text{N} &= \text{diag}\,(\nu_1 J_2, \nu_2 J_2, \ldots, \nu_{n/2} J_2) \text{ for } n \text{ even}\\
&= \text{diag}\,(\nu_1 J_2, \nu_2 J_2, \ldots, \nu_{(n-1)/2} J_2, 0) \text{ for } n \text{ odd}
\end{aligned} \tag{133}$$

in which the ν_j's are some (given) real numbers.

Taking $e = m - 1$ in (130) and considering the relation

$$N = \sum_{j=0}^{e} b_j G^{2j+1} \tag{134}$$

in which $m = n/2$ if n is even, and $m = (n-1)/2$ if n is odd, we get, upon premultiplication of (134) by Q^T and post-multiplication by Q,

$$\begin{aligned}
\text{N} = \text{diag}\,(\nu_1 J_2, \nu_1 J_2, \nu_m J_2, 0_{n-2m}) &= \sum_{l=0}^{m-1} b_l Q^T G^{2l+1} Q = \sum_{l=0}^{m-1} b_l \Gamma^{2l+1}\\
&= \sum_{l=0}^{m-1} (-1)^l b_l \text{diag}\,(\beta_1^{2l+1} J_2, \beta_2^{2l+1} J_2, \ldots, \beta_m^{2l+1} J_2, 0_{n-2m})
\end{aligned}$$

Equating the two quasi-diagonal matrices shown above and placing the numbers $\nu_1, \ldots, \nu_m$ into a column vector, we obtain

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \otimes J_2 = \begin{bmatrix} \beta_1 & -\beta_1^3 & \beta_1^5 & \cdots & (-1)^{m-1}\beta_1^{2m-1} \\ \beta_2 & -\beta_2^3 & \beta_2^5 & \cdots & (-1)^{m-1}\beta_2^{2m-1} \\ & \vdots & & & \\ \beta_m & -\beta_m^3 & \beta_m^5 & \cdots & (-1)^{m-1}\beta_m^{2m-1} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{bmatrix} \otimes J_2$$

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}}_{c} \otimes J_2 = \underbrace{\begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ & \beta_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \beta_m \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & \beta_1^2 & \cdots & \beta_1^{2m-2} \\ 1 & \beta_2^2 & \cdots & \beta_2^{2m-2} \\ & & \vdots & \\ 1 & \beta_m^2 & \cdots & \beta_m^{2m-2} \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} (-1)^0 & 0 & \dots & 0 \\ 0 & (-1)^1 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & (-1)^{m-1} \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{bmatrix}}_{b} \otimes J_2$$

where $\otimes$ denotes the Kronecker matrix product. Since $J_2^{-1} = -J_2$, upon multiplying the equation on the right by $-J_2$, it reduces to the relation $c = (BVE)b$ shown above. The matrix B is nonsingular since the β_j's are all nonzero; V is the m by m Vandermode matrix and it is nonsingular since the β_j's are distinct, i.e., $\beta_j \neq \beta_k$ for $j \neq k$; and, the diagonal matrix E is an m by m nonsingular matrix since, going down the diagonal, its elements successively alternate between the values 1 and -1. Hence, the determinant of each of the three matrices on the right-hand side is nonzero, and therefore the matrix product BVE is invertible. This implies that for any given column vector c, i.e., v_j, $j = 1, 2, \dots, m$, one can uniquely find the corresponding b_j's, $j = 0, 1, \dots, (m-1)$, that satisfy (134). We then have the following result.

Result 10. Every MDOF system described by (12) that can be simultaneously orthogonally quasi-diagonalized *must* have its matrix N expressible in the form (134) if the eigenvalues of $G \neq 0$ are distinct.

This then leads to the following result.

Result 11. If all eigenvalues of the n by n matrix K are distinct and the eigenvalues of G are distinct, then there exists a real orthogonal matrix Q such that (12) can be quasi-diagonalized and transformed by a real orthogonal coordinate transformation $x = Qp$ to the uncoupled form (109) if and only if

$$[K, S] = 0, \ [K, G^2] = 0, \text{ and } N = \sum_{j=0}^{m-1} b_j G^{2j+1} \tag{135}$$

where $m = n/2$ if n is even, and $(n-1)/2$ if n is odd. The uncoupled form leads to independent subsystems each having at most two degrees-of-freedom.

Proof. By Corollary 5 and Result 10, the result follows. ■

Example 5. Consider the MDOF system (12) described by the matrices

$$S = \begin{bmatrix} 0.2546 & 0.1037 & -0.0621 & -0.0833 \\ 0.1037 & 0.2586 & 0.0457 & -0.0086 \\ -0.0621 & 0.0457 & 0.2374 & 0.0339 \\ -0.0833 & -0.0086 & 0.0339 & 0.2494 \end{bmatrix}, \ G = \begin{bmatrix} 0 & -5 & -2 & -4 \\ 5 & 0 & -5 & -2 \\ 2 & 5 & 0 & -4 \\ 4 & 2 & 4 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 4 & 0.5 & 0 & -1 \\ 0.5 & 3 & -0.5 & -1 \\ 0 & -0.5 & 2 & 1 \\ -1 & -1 & 1 & 2 \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 2.56 & 2.32 & 1.40 \\ -2.56 & 0 & 2.56 & 2.32 \\ -2.32 & -2.56 & 0 & 1.40 \\ -1.40 & -2.32 & -1.40 & 0 \end{bmatrix}$$

and

$$f(t) = \begin{bmatrix} f_1(t) & f_2(t) & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

The spectrum of K is $\{0.7753, 2.00, 3.227, 5\}$, and the purely imaginary spectrum of G is $\{\pm 4.2426i, \pm 8.4853i\}$. Thus K and G both have distinct eigenvalues. A quick computation shows that $[K, S] = 0$. Furthermore,

$$KG^2 = \begin{bmatrix} -207 & -99 & 45 & 108 \\ -99 & -162 & -9 & 54 \\ 45 & -9 & -99 & -72 \\ 108 & 54 & -72 & -108 \end{bmatrix} = G^2 K$$

The orthogonal matrix Q made up of the eigenvectors of K, namely,

$$Q = \begin{bmatrix} -0.6739 & 0.2142 & -0.0000 & -0.7071 \\ 0.5502 & 0.1749 & -0.6667 & -0.4714 \\ -0.4266 & -0.5640 & -0.6667 & 0.2357 \\ -0.2473 & 0.7781 & -0.3333 & 0.4714 \end{bmatrix}$$

simultaneously quasi-diagonalizes all four matrices S, G, K, and N. Upon using the real orthogonal coordinate transformation $x = Qp$, two independent, two degrees-of-freedom subsystems are obtained.

They are

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 0.1 & 4.2426 \\ -4.2426 & 0.2 \end{bmatrix}\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} 3.2247 & -0.3394 \\ 0.3394 & 0.7753 \end{bmatrix} \times \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -0.6793 & 0.5502 \\ 0.2142 & 0.1749 \end{bmatrix}\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

and

$$\begin{bmatrix} \ddot{p}_3 \\ \ddot{p}_4 \end{bmatrix} + \begin{bmatrix} 0.3 & 8.4853 \\ -8.4853 & 0.4 \end{bmatrix}\begin{bmatrix} \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} + \begin{bmatrix} 2 & -5.2609 \\ 5.2609 & 5 \end{bmatrix} \times \begin{bmatrix} p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 0 & -0.6667 \\ -0.7071 & -0.4714 \end{bmatrix}\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

We note that N must have the form given in Result 11. Indeed, here $N = 0.1G + 0.01G^3$.

Remark 22. The roles of K and S and/or the roles of G and N can be interchanged in the above, starting from Result 9.

Remark 23. By using (11), all the results for uncoupling the MDOF system by simultaneous orthogonal quasi-diagonalization described by (12) can be translated to the system (1). The real nonsingular coordinate transformation $q(t) = \tilde{M}^{-1/2}Qp(t)$ will yield the quasi-diagonalized form (109), resulting in uncoupling (1) into independent two-degree-of-freedom subsystems. The eigenvalues of the matrices S, G, K, and N are identical to the eigenvalues of the matrices $\tilde{M}^{-1}\tilde{S}$, $\tilde{M}^{-1}\tilde{G}$, $\tilde{M}^{-1}\tilde{K}$, and $\tilde{M}^{-1}\tilde{N}$, respectively. For example, the commutation condition $[G, N] = 0$ becomes $[M^{-1/2}\tilde{G}M^{-1/2}, M^{-1/2}\tilde{N}M^{-1/2}] = 0$ or $\tilde{G}\tilde{M}^{-1}\tilde{N} = \tilde{N}\tilde{M}^{-1}\tilde{G}$, since $M^{-1/2}$ is nonsingular. Similarly, $[K, S] = 0$ becomes $\tilde{K}\tilde{M}^{-1}\tilde{S} = \tilde{S}\tilde{M}^{-1}\tilde{K}$. As an illustration, Result 11 translates to the following.

Result 12. If all eigenvalues of the n by n matrix $\tilde{M}^{-1}\tilde{K}$ are distinct and all eigenvalues of $\tilde{M}^{-1}\tilde{G}$ are distinct, then there exists a real orthogonal matrix Q such that (1) can be quasi-diagonalized and transformed by a real nonsingular coordinate transformation $q(t) = \tilde{M}^{-1/2}Qp(t)$ to the uncoupled form (109) if and only if

$$\tilde{K}\tilde{M}^{-1}\tilde{S} = \tilde{S}\tilde{M}^{-1}\tilde{K},\ \tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K}, \quad \text{and} \quad \tilde{N} = \sum_{j=0}^{m-1} b_j \tilde{G}(\tilde{M}^{-1}\tilde{G})^{2j} \tag{136}$$

where $m = n/2$ if n is even, and $(n-1)/2$ if n is odd. The uncoupled form leads to independent subsystems each having at most two degrees-of-freedom. ■

4 Conclusions

This article deals with the uncoupling of general linear MDOF systems with arbitrary stiffness and damping matrices into independent subsystems that have at most two degrees-of-freedom using simultaneous orthogonal quasi-diagonalization and real linear nonsingular coordinate transformations. The mass matrix of the MDOF system is assumed to be positive definite. The main results can be summarized as follows.

(1) A fundamental result in linear algebra that gives the necessary and sufficient (n&s) conditions for two symmetric n by n matrices (S and K) and two skew-symmetric n by n matrices (G and N) to be simultaneously quasi-diagonalized by a real orthogonal matrix is developed. It is shown that there are a total of 16 commutation conditions that are n&s for their simultaneous orthogonal quasi-diagonalization. The commutation conditions involve only these four matrices.

(2) The mass matrix of an n-degree-of-freedom system is normalized to the identity matrix I, and the damping matrix, $D = S + G$, is split into its symmetric (S) and skew-symmetric (G) additive parts; likewise, the stiffness matrix, $R = K + N$, is also split into its symmetric (K) and skew-symmetric (N) parts. The result in (1) above is then used to obtain the n&s conditions for the simultaneous orthogonal quasi-diagonalization of the four matrices S, G, K, and N. These n&s conditions, when satisfied, maximally uncouple a general MDOF dynamical system, through the use of a real linear orthogonal coordinate transformation, into independent subsystems, each with at most two degrees-of-freedom. The structure of the matrices of the resulting uncoupled two-degree-of-freedom subsystems is obtained. A procedure for the explicit determination of the orthogonal matrix Q that accomplishes this simultaneous quasi-diagonalization, when the n&s conditions are satisfied, is provided.

(3) The n&s conditions obtained are shown to provide a unified way of determining the n&s conditions for the maximal uncoupling, through simultaneous orthogonal quasi-diagonalization, of various categories of linear MDOF dynamical systems, such as gyroscopic potential systems, potential systems with arbitrary damping matrices, etc. Consequently, the article presents a general set of n&s conditions from which earlier results in Refs. [1,3–7] can be directly obtained. These results pertain to the uncoupling of different categories of structural and mechanical MDOF systems [9] into independent subsystems of at most two degrees-of-freedom through simultaneous quasi-diagonalization.

(4) The matrix quadruple $\{S, G, K, N\}$ can be simultaneously orthogonally quasi-diagonalized if and only if the four matrix triplets $\{S, G, K\}$, $\{S, N, K\}$, $\{S, G, N\}$, and $\{K, G, N\}$, with $G \neq 0$, $N \neq 0$, and $G(N) \neq 0$, and $G(N) \neq 0$, respectively, can individually be orthogonally quasi-diagonalized.

(5) When the rank, $2m$, of the skew-symmetric matrix $G(N)$ is such that the number of degrees-of-freedom, n, of the MDOF system exceeds the rank of $G(N)$ by at most 2 (i.e., $n \leq 2m + 2$)—a common occurrence in many MDOF dynamical systems found in nature and in engineered systems—the number of n&s conditions to uncouple the system, in the manner stated above, through the use of a real orthogonal coordinate transformation drops from 16 in the general case when $n > 2m + 2$ to ten.

(6) Noting that the 16 n&s conditions, though reduced to ten, still constitute significant restrictions on the matrices that describe a linear MDOF system, the n&s conditions are further reduced when the results are applied to real-life systems encountered in nature and in aerospace, civil, and mechanical engineering. When $n > 2m + 2$ and the nonzero eigenvalues of G as well as the nonzero eigenvalues of N are distinct, or, when the eigenvalues of $K(S)$ are distinct, the number of n&s conditions for uncoupling the system, by simultaneous orthogonal quasi-diagonalization, reduce from 16 to 6; when the eigenvalues of $K(S)$ are distinct and the nonzero eigenvalues of $G(N)$ are distinct, they reduce to 5; and, when the eigenvalues of $K(S)$ are distinct, the nonzero eigenvalues of G are distinct, and the nonzero eigenvalues of N are distinct—all generic properties of these matrices—they reduce from 16 to 4.
Furthermore, by positing a structure on the matrices $S(K)$ and $N(G)$, it is shown that the number of n&s conditions for uncoupling such MDOF dynamical systems reduce to just 2, and they further reduce to a single n&s condition when, in addition, the nonzero eigenvalues of $G(N)$ are distinct. Also, when the eigenvalues of $K(S)$ and the nonzero eigenvalues of $G(N)$ are distinct, two n&s conditions for uncoupling such MDOF systems are obtained. It is also shown that when the eigenvalues of $K(S)$ are distinct and the eigenvalues of $G(N)$ are distinct, two n&s conditions are required

for the simultaneous orthogonal quasi-diagonalization of the MDOF system, and furthermore, the matrix $N(G)$ must have the posited form.

(7) When the number of degrees-of-freedom of the MDOF system exceeds the rank of the skew-symmetric matrix $G(N)$ by at most 2 ($n \leq 2m+2$), a common occurrence in many structural and mechanical systems, it is shown that the number of n&s conditions for simultaneous quasi-diagonalization drops from 16 to 10. Furthermore, if the nonzero eigenvalues of G and the nonzero eigenvalues of N are distinct, or, when the eigenvalues of $K(S)$ are distinct, the number of n&s conditions for uncoupling the MDOF system, by quasi-diagonalization, reduce to 4; and, when the eigenvalues of $K(S)$ are distinct and the nonzero eigenvalues of $G(N)$ are distinct, they reduce to just 3.

Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

No data, models, or code were generated or used for this paper.

Appendix

Proof of Result 9. We compute the remaining 14 commutators in (91) when $[K, G^2] = [K, GKG] = 0$ and show that they all equal zero for $S = \sum_{j=0}^{n-1} a_j K^j$ and $N = \sum_{j=0}^{e} b_j G^{2j+1}$ where the a_j's and b_j's are arbitrary real constants, and e is a non-negative integer.

Lemma 10(c) says that $[K^j, G^{2l+1}K^u G^{2r+1}] = 0$ for any non-negative integers j, l, u, and r when $[K, G^2] = [K, GKG] = 0$.

The remaining 14 commutators yield the following results:

$$[G, N] = \left[G,\ \sum_{j=0}^{e} b_j G^{2j+1}\right] = \sum_{j=0}^{e} b_j [G, G^{2j+1}] = 0$$

$$[K, S] = \left[K,\ \sum_{j=0}^{n-1} a_j K^j\right] = \sum_{j=0}^{n-1} a_j [K, K^j] = 0$$

$$[K, GN] = \left[K,\ \sum_{j=0}^{e} b_j G^{2j+2}\right] = \sum_{j=0}^{e} b_j [K, G^{2j+2}] = 0$$

$$[K, N^2] = \left[K,\ \sum_{k=0}^{e}\sum_{j=0}^{e} b_j b_k G^{2k+1} G^{2j+1}\right] = \sum_{k=0}^{e}\sum_{j=0}^{e} b_j b_k [K, G^{2k+2j+2}] = 0$$

$$[K, NKN] = \left[K,\ \sum_{j=0}^{e} b_j G^{2j+1} \sum_{k=0}^{e} b_k K G^{2k+1}\right] = \left[K,\ \sum_{j=0}^{e}\sum_{k=0}^{e} b_j b_k G^{2j+1} K G^{2k+1}\right] = \sum_{j=0}^{e}\sum_{k=0}^{e} b_j b_k \left[K, G^{2j+1} K G^{2k+1}\right] = 0$$

$$[K, GSG] = \left[K,\ G\left(\sum_{j=0}^{n-1} a_j K^j\right) G\right] = \sum_{j=0}^{n-1} a_j [K, GK^j G] = 0$$

$$[S, GKG] = \left[\sum_{j=0}^{n-1} a_j K^j,\ GKG\right] = \sum_{j=0}^{n-1} a_j [K^j, GKG] = 0,$$

by Remark 4, part 4

$$[S, G^2] = \left[\sum_{j=0}^{n-1} a_j K^j,\ G^2\right] = \sum_{j=0}^{n-1} a_j [K^j, G^2] = 0$$

$$[S, N^2] = \left[\sum_{j=0}^{n-1} a_j K^j,\ N^2\right] = \sum_{j=0}^{n-1} a_j [K^j, N^2] = 0$$

$$[S, GSG] = \left[\sum_{j=0}^{n-1} a_j K^j,\ G\left(\sum_{l=0}^{n-1} a_l K^l\right) G\right] = \sum_{j=0}^{n-1}\sum_{l=0}^{n-1} a_j a_l [K^j, GK^l G] = 0,$$

by Lemma (10 c)

$$[S, GN] = \left[\sum_{j=0}^{n-1} a_j K^j,\ GN\right] = \sum_{j=0}^{n-1} a_j [K^j, GN] = 0$$

$$[S, NKN] = \left[\sum_{j=0}^{n-1} a_j K^j,\ NKN\right] = \sum_{j=0}^{n-1} a_j [K^j, NKN] = 0$$

$$[K, NSN] = \left[K,\ \sum_{l=0}^{e} b_l G^{2l+1} \sum_{u=0}^{n-1} a_u K^u \sum_{r=0}^{e} b_r G^{2r+1}\right] = \sum_{j=0}^{n-1}\sum_{l=0}^{e}\sum_{r=0}^{e} a_j b_l b_r [K, G^{2l+1} K^u G^{2r+1}] = 0,$$

by Lemma 11, and

$$[S, NSN] = \left[\sum_{j=0}^{n-1} a_j K^j,\ NSN\right] = \sum_{j=0}^{n-1} a_j [K^j, NSN] = 0,$$

by Remark 4, part 4.

This proves the result. ■

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